Case $s_1 = s_2$

ODE: $R(x)\frac{d^2y}{dx^2} + \frac{1}{x}P(x)\frac{dy}{dx} + \frac{1}{x^2}Q(x)y = 0 \qquad = \mathcal{L}y$

$$\mathcal{L} = R\frac{d^2}{dx^2} + P\frac{1}{x}\frac{d}{dx} + \frac{1}{x^2}Q(x)$$

$y(x,s) \equiv x^s \sum_{k=0}^{\infty} A_k x^k$ {if $\mathcal{L}y=0$, then $s = s_1$ or s_2, y: solution} afterthought

$\mathcal{L}(x,s) = A_0 f(s) x^{s-2} + \dots$

$\quad \llcorner f(s) = (s-s_1)^2$

(because $s=s_2$) $\qquad (A_0 \neq 0)$

- $y(x,s)$ is the solution of the ODE when $\mathcal{L}y(x,s)=0 \rightarrow s = s_1$ coef. of $x^{s-2} = 0$
- $\frac{d}{ds}\mathcal{L}y(x,s)\Big|_{s=s_1} = A_0 \underbrace{f'(s)}_{2(s-s_1)} x^{s-2} + \dots \Big|_{s=s_1} = 0$

$\frac{d}{ds}\mathcal{L}y(x,s)|_{s=s_1} = 0 \iff \mathcal{L}\frac{d}{ds}y(x,s)|_{s=s_1} = 0$

$\rightarrow \frac{d}{ds} = y(x,s)_{s=s_1}$; solution of ODE $\mathcal{L}y=0$

$y(x,s) = \sum_{n=0}^{\infty} A_k(s) x^{k+s} \rightarrow e^{(k+s)\ln x} \qquad x>0$

$\qquad \rightarrow \frac{d}{ds} e^{(k+s)\ln x} = \ln x \cdot x^{k+x}$

$\frac{dy}{ds}\Big|_{s=s_1} = \sum_{n=0}^{\infty} A_k'(s_1) x^k + \ln x \cdot \underbrace{x^{s_1} \cdot \sum_{k=0}^{\infty} A_k^{(s_1)} x^k}_{y_1(x) = Cy(x,s_1)}$

<u>$s = s_1 = s_2$</u>: 1 solution is $y_1(x) = x^{s_1}\sum_{k=0}^{\infty} A_k x^k$

2nd solution: $y_2(x) = \sum_{k=0}^{\infty} B_k x^{k+s_1} + C \ln x \, y_1(x)$ ← any y_1

$(B_k = A_k')$

find B_k and C from ODE (by direct substitution)

"Particular type" of ODE:

$$(1+R_n x^m)\, y'' + \frac{1}{x}(P_0 + P_m x^m)\, y' + \frac{1}{x^2}(Q_0 + Q_m x^m)\, y = 0$$

m: integer >0

indicial equation: $s(s-1)+P_0 s+Q_0=0 \rightarrow$ 2 roots s_1, s_2

$g_n(s) = R_n(s-n)(s-n-1)+P_0(s-n)+Q_n$, $n\ge 1$

$R_n, P_n, Q_n = 0$, $n \neq M, 0$

$g_n(s) = 0$, $n \neq M$, $n \geq 1$

$g_m(s) = R_M(s-m)(s-m-1)+P_m(s-M)+Q_M$

$$y = x^s \sum_{k=0}^{\infty} A_k x^k \xrightarrow{ODE} \quad f(s)=0,\ A_0 \neq 0$$

$$f(s+k)A_k + \underbrace{\sum_{n=1}^{k} g_n(s+k)A_{k-n}}_{\substack{0,\ \text{if } 1\le k<m \\ g_m(s+k)A_{k-m}\ \text{if } k\ge m}} = 0$$

Recurrence relations ($A_0 \neq 0$), $s=s_1$ or s_2:

$1 \le k < m$: $f(s+k)A_k = 0 \rightarrow A_k = 0$, $k=1,\dots,m-1$

$k \ge m$: $f(s+k)A_k + g_m(s+k)\cdot A_{k-m} = 0$ $\qquad A_k = \frac{-g(s+k)}{f(s+k)}\cdot A_{k-m}$; $f(s+k)\neq 0$

$g_m(s) \equiv g(s)$

$A_1, A_2, \dots, A_{m-1} = 0 \quad A_m \neq 0$, in terms of A_0

$A_{m+1}, \dots, A_{2m-1} = 0 \quad A_{2m} \neq 0$, in terms of A_0

$\vdots$

all coefficients are zero except $A_{lm} \neq 0$ l=integer >0

$$y(x) = x^s \sum_{k=0}^{\infty} A_k x^k = x^s \sum_{l=0}^{\infty} B_l x^{lm}$$

$A_k = 0$ if $k \neq$ multiple of m

$B_l \equiv A_{lm}$