ex find C_{-1} for $f(z) = z e^{1/z}$ w $\quad z_0 = 0$

$$f(z) = z\left[1 + \frac{1}{z} + \frac{1}{2!}\cdot\frac{1}{z^2} + \dots + \frac{1}{n!}\frac{1}{z^n} + \dots\right] = \frac{1}{n!}\frac{1}{z^{n-1}}$$

$C_{-1}: \frac{1}{2!}\frac{1}{z}$ residue: $\frac{1}{2}$

Definition: $C_{-1} = \underset{z=z_0}{\mathrm{Res}}\, f(z)$

Theorem:

$$\boxed{\begin{array}{l} \text{If } f(z) = \dfrac{g(z)}{h(z)},\ g,h: \text{analytic in } 0 \le |z-z_0| < \delta \\ \qquad h(z_0) = 0,\ h'(z_0) \ne 0 \\ \text{then } \underset{z=z_0}{\mathrm{Res}}\, f(z) = \dfrac{g(z_0)}{h'(z_0)} \quad (z_0 = \text{simple pole}) \end{array}}$$

$$f(z) = \frac{g(z_0) + (z-z_0)g'(z_0) + \dots}{h(z_0) + (z-z_0)h'(z_0) + \dots} \quad \begin{array}{l}\text{(Taylor series for } g(z))\\ \text{(Taylor series for } h(z))\end{array}$$

$$= \frac{1}{z-z_0}\,\frac{g(z_0)}{h'(z_0)}\,\frac{1 + \frac{g'(z_0)}{g(z_0)}(z-z_0) + \dots}{1 + \frac{1}{2}\frac{h''(z_0)}{h'(z_0)} + \dots}$$

analytic $0 \le |z-z_0| < \delta$

$= 1$ at $z = z_0$

$$f(z) = \boxed{\frac{g(z_0)}{h'(z_0)}} \cdot \frac{1}{z-z_0} + (\text{non-negative powers of } (z-z_0))$$

↳ $C_{-1} \ne 0$ if $g(z_0) \ne 0$

Theorem:

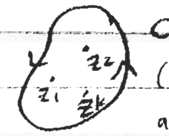
$$\boxed{\begin{array}{l} \text{If } z_0: m^{th} \text{ order pole of } f(z), \\ C_{-1} = \displaystyle\lim_{z \to z_0} \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\left[(z-z_0)^m f(z)\right] \end{array}}$$

useful when $m = 1$ or 2

Residue Theorem

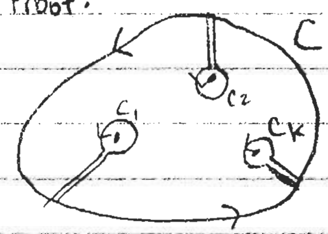
If $f(z)$ has isolated singularities at $z_1, z_2, \dots, z_n$ and is analytic elsewhere, then

$$\oint_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z)$$



(C encloses all these points counterclockwise)

Proof:



$C' = C + C_1 + C_2 + \dots + C_k$

by Cauchy Integral Theorem,

$$\oint_{C'} f(z)\,dz = \left[\oint_C - \oint_{C_1} - \oint_{C_2} - \dots - \oint_{C_k}\right] f(z)\,dz = 0$$

$$\therefore \oint_C f(z)\,dz = \left[\oint_{C_1} + \oint_{C_2} + \oint_{C_3}\right] f(z)\,dz$$

$$\oint_C f(z)\,dz = \sum_{k=1}^{n} \underbrace{\oint_{C_k} f(z)\,dz}_{2\pi i \operatorname*{Res}_{z=z_k} f(z)}$$ ← isolated singularities of $f(z)$ within C

$$\boxed{\oint_C f(z)\,dz = \sum_{k=1}^{n} 2\pi i \operatorname*{Res}_{z=z_k} f(z)}$$