18.075 Solutions to Quiz 3

(I)

ODE : $xy'' - xy' - y = 0$ (1)

(1) We rewrite this ODE in the form

$$y'' - y' - \frac{1}{x} y = 0 \quad , \quad \text{i.e.,} \quad y'' + a_1(x) y' + a_2(x) y = 0 ,$$

where

$$a_1(z) = -1 \quad , \quad a_2(z) = -\frac{1}{z} .$$

In particular, $a_2(z)$ is NOT analytic at $z = 0 = x_0$.

Hence, $x_0 = 0$ is a SINGULAR point.

Since $z^2 a_2(z) = -z$: analytic , $x_0 = 0$ is a REGULAR singular point.

(2) ODE: $y'' + \frac{1}{x}(-x) y' + \frac{1}{x^2}(-x) y = 0$.

So,

$$R(x) \equiv 1 , \quad P(x) \equiv -x \quad , \quad Q(x) \equiv -x .$$

(3) Indicial equation:

$$f(s) = 0 \iff s(s-1) + P_0 s + Q_0 = 0 \quad \text{where} \quad P_0 = 0, \quad Q_0 = 0$$

$$\iff s(s-1) = 0 \iff s = 0, 1 \; ; \quad \boxed{s_1 = 1, \; s_2 = 0}$$

(taking the largest of 2 real roots to be s_1)

④ Frobenius method: $y(x) = x^s \sum_{k=0}^{\infty} A_k x^k$, $s = s_1 = 1$ or $s = s_2 = 0$; $\boxed{A_0 \neq 0}$

Recursive function g_n : $g_n(s) = R_n(s-n)(s-n-1) + P_n(s-n) + Q_n$, $n \geq 1$.

$\Rightarrow \{ g_n(s) = 0$ for $n \neq 1$, and $g_1(s) = P_1(s-1) + Q_1 = -s+1-1 = -s \}$.

Recurrence formula : $f(s+k) A_k = -\sum_{n=1}^{k} g_n(s+k) A_{k-n}$, $k = 1, 2, \ldots$,

Take $s = s_1 = 1$: $f(1+k) A_k = -g_1(1+k) A_{k-1}$

$$\Rightarrow k\cancel{(1+k)} A_k = +\cancel{(1+k)} A_{k-1} \Leftrightarrow A_k = +\frac{A_{k-1}}{k}, \quad k = 1, 2, 3, \ldots$$

$$\left.\begin{array}{ll} \underline{k=1}: & A_1 = \frac{A_0}{1} \\ \underline{k=2}: & A_2 = A_1/2 \\ \underline{k=3}: & A_3 = A_2/3 \\ \vdots & \\ \underline{k=k}: & A_k = \frac{A_{k-1}}{k} \end{array}\right\} \xRightarrow{\text{multiply}} A_k = \frac{A_0}{1 \cdot 2 \cdot 3 \ldots k} = \frac{A_0}{k!}$$

$$y(x) = y_1(x) = x^{s_1} \sum_{k=0}^{\infty} A_k x^k = x^1 \sum_{k=0}^{\infty} \frac{A_0}{k!} x^k = A_0 x \sum_{k=0}^{\infty} \frac{x^k}{k!} = A_0 x e^x \equiv A_0 u_1(x)$$

where $\underline{u_1(x) = x e^x}$.

⑤ Take $s = s_2 = 0$:

Recurrence formula: $f(s_2+k) A_k = -g_1(s_2+k) A_{k-1}$ $k = 1, 2, 3, \ldots$

$$\Rightarrow \cancel{k}(k-1) A_k = \cancel{k} A_{k-1} \Rightarrow (k-1) A_k = A_{k-1}.$$

Check the formula for $\underline{k=1}$: $0 = 0 \cdot A_1 = A_0 \neq 0$ $\Rightarrow$ impossible to find a solution!

Hence, we cannot find any independent solution by this method if $s = s_2 = 0$.

⑥ A second independent solution is of the form:

$$y_2(x) = C_1 \underbrace{x e^x}_{u_1(x)} \ln x + \sum_{k=0}^{\infty} B_k x^{k+s_2}, \quad C \neq 0 : \text{arbitrary}.$$

General solution: $y(x) = y_1(x) + y_2(x)$, A_0, C: arbitrary.

II. Bessel ODE: $x^2y'' + xy' + (x^2-p^2)y = 0.$

1. $\boxed{p=1/2}$:

$$y(x) = C_1 J_{1/2}(x) + C_2 J_{-1/2}(x)$$

$$= C_1\sqrt{\frac{2}{\pi x}}\sin x + C_2\sqrt{\frac{2}{\pi x}}\cos x .$$

2. (cont.)

$$0 = y(\pi) = C_1\sqrt{\frac{2}{\pi^2}}\cancel{\sin\pi}^{\,0} + C_2\sqrt{\frac{2}{\pi\pi}}\cos\pi = C_2\frac{\sqrt{2}}{\pi}(-1) \Rightarrow \boxed{C_2=0}$$

Then $y(x) = C_1\sqrt{\frac{2}{\pi x}}\sin x \Rightarrow y'(x) = C_1\sqrt{\frac{2}{\pi x}}\cos x - \frac{C_1}{2x}\sqrt{\frac{2}{\pi x}}\sin x$

$$\Rightarrow y'(\pi) = C_1\frac{\sqrt{2}}{\pi}\cos\pi - \frac{C_1}{2\pi}\sqrt{\frac{2}{\pi^2}}\cancel{\sin\pi}^{\,0} = -C_1\frac{\sqrt{2}}{\pi} = 1$$

$$\Rightarrow \boxed{C_1 = -\frac{\pi}{\sqrt{2}}}$$

Hence,

$$y(x) = -\sqrt{\frac{\pi}{x}}\sin x$$

3. $\boxed{p=2}$: $y(x) = c_1 J_2(x) + c_2 Y_2(x)$

4. Limiting behaviors: $J_p(x) \approx \frac{1}{2^p p!}x^p$ as $x\to 0$

$$Y_p(x) \approx -\frac{2^p(p-1)!}{\pi}x^{-p} \text{ as } x\to 0$$

For $\underline{p=2}$: $J_2(x) \simeq \frac{1}{8}x^2$ as $x\to 0$

$$Y_2(x) \simeq -\frac{4}{\pi}x^{-2} \text{ as } x\to 0.$$

It follows that $J_2(0)=0$ while $Y_2(x)$ "blows up" as $x\to 0$.

Hence, in order to have $\lim_{x\to 0} y(x) \neq \infty$, we must take $\underline{c_2=0}$.

Then $y(x) = c_1 J_2(x) \Rightarrow \lim_{x\to 0} y(x) = c_1 J_2(0) = c_1\cdot 0 = 0$ for any c_1.

Hence, $\boxed{y(x) = c_1 J_2(x)}$. (not unique).

III ① ODE: $xy''-9y'+xy=0 \Rightarrow x^2y''+x(-9)y'+x^2y=0$

We try to identify this equation with the form

$$x^2y''+x[(1-2A)+2rBx^r]y'+[A^2-p^2s^2+s^2C^2x^{2s}-rB(2A-r)x^r+r^2B^2x^{2r}]y=0$$

which has the solution $y(x)=x^A e^{-Bx^r} Z_p(Cx^s)$.

Coef. of y': $x\cdot\ -9 \equiv x\cdot[(1-2A)+2rBx^r]$

Clearly, $1-2A=-9 \Leftrightarrow \boxed{A=5}$, $\boxed{B=0}$

Coef. of y: $x^2 \equiv A^2-p^2s^2+s^2C^2x^{2s}-r\cancel{B}^{\,0}(2A-r)x^r+r^2\cancel{B}^{\,0\,2}x^{2r}$

It follows that: $2s=2 \Leftrightarrow \boxed{s=1}$ $C^2=1 \Leftrightarrow \boxed{C=1}$

$A^2-p^2s^2=0 \Leftrightarrow A^2=p^2s^2 \Leftrightarrow p^2=\frac{A^2}{s^2}=25 \Leftrightarrow \boxed{p=5}$ $(p\geq0)$

Solution: $y(x)=x^5e^0 Z_5(x) = \boxed{x^5\ [C_1J_5(x)+C_2Y_5(x)]}$

② ODE: $xy''+(1+6x^2)y'+x(2+9x^2)y=0 \Rightarrow x^2y''+x(1+6x^2)y'+x^2(2+9x^2)y=0$

Coef. of y': $x\cdot(1+6x^2) \equiv x\cdot[(1-2A)+2rBx^r]$

$1=1-2A \Leftrightarrow \boxed{A=0}$, $\boxed{r=2}$, $2rB=6 \Leftrightarrow 4B=6 \Leftrightarrow \boxed{B=3/2}$

Coef. of y: $[x^2(2+9x^2)=2x^2+9x^4] \equiv A^2-p^2s^2+s^2C^2x^{2s}-rB(2A-r)x^r+r^2B^2x^{2r}$

$r^2B^2x^{2r}=4\cdot\frac{9}{4}x^4=9x^4$, as it should be

$rB(2A-r)x^r=2\cdot 3/2\ (2\cdot0-2)x^2=-6x^2$, whereas the term in LHS is $2x^2$.

Hence, we must take $2s=2 \Leftrightarrow \boxed{s=1}$

$s^2C^2x^{2s}-rB(2A-r)x^r=2x^2 \Leftrightarrow (C^2+6)x^2=2x^2 \Leftrightarrow C^2=-4 \Leftrightarrow \boxed{C=2i}$

$$A^2 - p^2 s^2 = 0 \quad \Leftrightarrow \quad p^2 = A^2/s^2 = 0 \quad \Leftrightarrow \quad \boxed{p=0}$$

Solution: $$y(x) = x^0 \, e^{-\frac{2}{3}x^2} Z_0(2ix) = \boxed{e^{-\frac{2}{3}x^2}\left[c_1 J_0(2ix) + c_2 Y_0(2ix)\right]}$$

Alternatively, $$y(x) = e^{-\frac{2}{3}x^2}\left[\tilde{c}_1 \cdot I_0(2x) + \tilde{c}_2 \cdot K_0(2x)\right],$$

where $I_0(z)$ and $K_0(z)$ are modified Bessel functions.

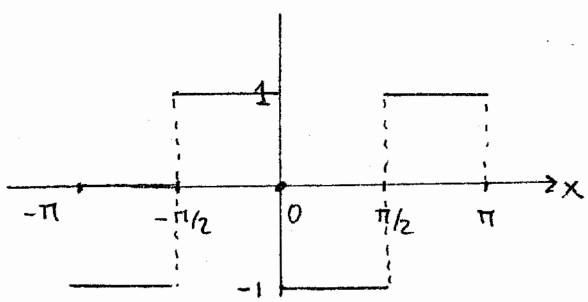
IV 1. Require $$h(x) = \sum_{n=1}^{\infty} B_n \cdot \sin(nx), \qquad 0 < x < \pi = \ell.$$

$$B_n = \frac{2}{\ell}\int_0^{\ell} dx \cdot h(x) \cdot \sin(nx) = \frac{2}{\pi}\left[\int_0^{\pi/2} dx\,(-1)\sin(nx) + \int_{\pi/2}^{\pi} dx \cdot \sin(nx)\right]$$
$(\ell = \pi)$

$$= \frac{2}{\pi}\left[\frac{1}{n}\cos(nx)\Big|_0^{\pi/2} - \frac{1}{n}\cos(nx)\Big|_{\pi/2}^{\pi}\right]$$

$$= \frac{2}{\pi}\left[\frac{1}{n}\left(\cos\frac{n\pi}{2} - 1\right) - \frac{1}{n}\left(\cos n\pi - \cos\frac{n\pi}{2}\right)\right] \quad ; \quad \cos(n\pi) = (-1)^n$$

$$= \frac{2}{\pi}\left[\frac{(-1)^{n+1}}{n} + \frac{2}{n}\cos\frac{n\pi}{2} - \frac{1}{n}\right] = \frac{2}{\pi n}\left[(-1)^{n+1} + 2\cos\frac{n\pi}{2} - 1\right] = \begin{cases} 0, & n: \text{odd} \\ \frac{2}{n\pi}\left[(-1)^{n/2} - 1\right], & n: \text{even.} \end{cases}$$



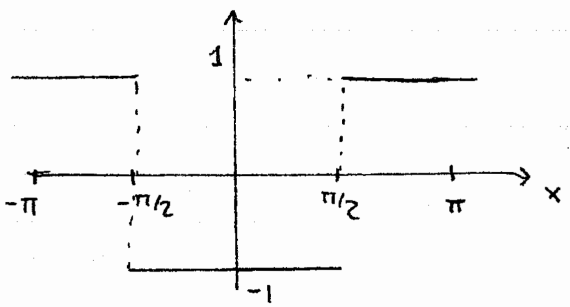
② $$h(x) = A_0 + \sum_{n=1}^{\infty} A_n \cdot \cos(nx)$$

$$A_0 = \frac{1}{\pi}\int_0^{\pi} dx \cdot h(x) = \frac{1}{\pi}\left(-\int_0^{\pi/2} dx + \int_{\pi/2}^{\pi} dx\right) = 0.$$

$$A_n = \frac{2}{\pi}\int_0^{\pi} dx \cdot h(x)\cdot\cos(nx) = \frac{2}{\pi}\left[-\int_0^{\pi/2} dx \cdot \cos(nx) + \int_{\pi/2}^{\pi} dx \cdot \cos(nx)\right]$$

$$= \frac{2}{\pi}\left[-\frac{1}{n}\sin(nx)\Big|_0^{\pi/2} + \frac{1}{n}\sin(nx)\Big|_{\pi/2}^{\pi}\right] \quad ; \qquad \sin(n\pi) = 0.$$

$$= \frac{2}{\pi}\left(-\frac{1}{n}\sin\frac{n\pi}{2} - \frac{1}{n}\sin\frac{n\pi}{2}\right) = -\frac{4}{n\pi}\sin\frac{n\pi}{2} = \begin{cases} 0, & n: \text{even} \\ \frac{4}{n\pi}(-1)^{\frac{n+1}{2}}, & n: \text{odd} \end{cases}$$



③ ODE : $\quad y''(x) + p^2 y = h(x), \qquad 0 < x < \pi.$

BC's : $\quad y(0) = y(\pi) = 0$

Because of the given bc's we try $y(x) = \sum_{n=1}^{\infty} B_n \cdot \sin(nx)$ and find B_n by substitution into the ODE; this series for y satisfies both BC's.

$$y'(x) = \sum_{n=1}^{\infty} +nB_n\cos(nx), \quad y''(x) = \sum_{n=1}^{\infty} -n^2 B_n \sin(nx)$$

ODE : $$-\sum_{n=1}^{\infty} n^2 B_n \sin(nx) + p^2\sum_{n=1}^{\infty} B_n \cdot \sin(nx) = h(x) \Leftrightarrow \sum_{n=1}^{\infty} \underbrace{(p^2-n^2)B_n}_{C_n}\sin(nx) = h(x) \quad Ⓐ$$

It follows that C_n are the coefficients found in part ① for h(x):

$$C_n = \begin{cases} 0, & n: \text{odd} \\ \frac{2}{n\pi}\cdot\left[(-1)^{n/2} - 1\right], & n: \text{even} \end{cases} = (p^2-n^2)\cdot B_n \Leftrightarrow B_n = \begin{cases} 0, & n: \text{odd} \\ \frac{2}{(p^2-n^2)n\pi}\left[(-1)^{n/2}-1\right], & n: \text{even} \end{cases} \text{ if } \boxed{p \neq \text{pos. integer}}$$

Suppose that $p =$ positive integer $= m$.

From ODE (Eq. Ⓐ) : $\sum_{n=1}^{\infty}(p^2-n^2)\, B_n \cdot \sin(nx) = \underbrace{\sum_{n=1}^{\infty} C_n \cdot \sin(nx)}_{h(x)}$ ← coeffs. for $h(x)$

$\Leftrightarrow$ $(p^2-n^2)B_n = C_n$ for all $n = 1, 2, \ldots$

$\Leftrightarrow$ $(m^2-n^2)B_n = C_n$.

For $n \neq m$: $B_n = \dfrac{C_n}{m^2-n^2}$, as before.

For <u>$n = m$</u>: $0 \cdot B_m = C_m$. This equation can NOT be true

UNLESS $C_m = 0 \Leftrightarrow$ ($m =$ odd or $m =$ multiple of 4)

So, the sine series expansion yields a solution only when

$p \neq$ positive integer or $p =$ odd integer or $p =$ multiple of $4 = 4k$ $(k = 1, 2, \ldots)$.

In all other cases for p the sine series fails to give a solution

In the case where $p =$ integer $=m$ gives a solution, B_m

in the sine series for $y(x)$ is NOT unique ; it is arbitrary.

So, then $y(x)$ given by the sine series is then <u>not</u> a unique solution

of the ODE.