SOLUTION SET I FOR 18.075–FALL 2004

10. Functions of a Complex Variable

10.1. **Introduction. The Complex Variable.** .

3. Establish the following results:
(a) $\text{Re}(z_1 + z_2) = \text{Re}(z_1) + \text{Re}(z_2)$, but $\text{Re}(z_1 z_2) \neq \text{Re}(z_1)\text{Re}(z_2)$ in general;
(b)$\text{Im}(z_1 + z_2) = \text{Im}(z_1) + \text{Im}(z_2)$, but $Im(z_1 z_2) \neq \text{Im}(z_1)\text{Im}(z_2)$ in general;
(c)$|z_1 z_2| = |z_1||z_2|$, but $|z_1 + z_2| \neq |z_1| + |z_2|$ in general;
(d)$\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$ and $\overline{z_1 z_2} = \overline{z_1 z_2}$.

Solution. (a) We want to show that $\text{Re}(z_1 + z_2) = \text{Re}(z_1) + \text{Re}(z_2)$. Let $z_1 = a_1 + ib_1$, $z_2 = a_2 + ib_2$, then

$$z_1 + z_2 = (a_1 + a_2) + i(b_1 + b_2).$$

Hence

$$\text{Re}(z_1 + z_2) = a_1 + a_2$$

and clearly

$$\text{Re}(z_1) + \text{Re}(z_2) = a_1 + a_2.$$

Let us show that in general $\text{Re}(z_1 z_2) \neq \text{Re}(z_1)\text{Re}(z_2)$. We have

$$z_1 z_2 = (a_1 + ib_1)(a_2 + ib_2) = (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1),$$

therefore

$$\text{Re}(z_1 z_2) = a_1 a_2 - b_1 b_2.$$

On the other hand

$$\text{Re}(z_1)\text{Re}(z_2) = a_1 a_2 \neq a_1 a_2 - b_1 b_2$$

in general.

(b) We want to show that $\text{Im}(z_1 + z_2) = \text{Im}(z_1) + \text{Im}(z_2)$. From part (a) we have that

$$z_1 + z_2 = (a_1 + a_2) + i(b_1 + b_2).$$

Hence

$$\text{Im}(z_1 + z_2) = b_1 + b_2$$

and clearly

$$\text{Im}(z_1) + \text{Im}(z_2) = b_1 + b_2.$$

Let us show that in general $\text{Im}(z_1 z_2) \neq \text{Im}(z_1)\text{Im}(z_2)$.We have

$$z_1 z_2 = (a_1 + ib_1)(a_2 + ib_2) = (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1),$$

therefore

$$\text{Im}(z_1 z_2) = a_1 b_2 + a_2 b_1.$$

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On the other hand

$$\mathrm{Im}(z_1)\mathrm{Im}(z_2) = b_1 b_2 \neq a_1 b_2 + a_2 b_1$$

in general.

(c) We want to show that $|z_1 z_2| = |z_1||z_2|$. From part (a) we have

$$z_1 z_2 = (a_1 + ib_1)(a_2 + ib_2) = (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1).$$

Hence

$$\begin{aligned} |z_1 z_2| &= \sqrt{(a_1 a_2 - b_1 b_2)^2 + (a_1 b_2 + a_2 b_1)^2} \\ &= \sqrt{(a_1^2 a_2^2 - 2a_1 a_2 b_1 b_2 + b_1^2 b_2^2) + (a_1^2 b_2^2 + 2a_1 b_2 a_2 b_1 + a_2^2 b_1^2)} \\ &= \sqrt{a_1^2 a_2^2 + b_1^2 b_2^2 + a_1^2 b_2^2 + a_2^2 b_1^2}. \end{aligned}$$

On the other hand $|z_1| = \sqrt{a_1^2 + b_1^2}$ and $|z_2| = \sqrt{a_2^2 + b_2^2}$. Therefore

$$|z_1||z_2| = (\sqrt{a_1^2 + b_1^2})(\sqrt{a_2^2 + b_2^2}) = \sqrt{(a_1^2 + b_1^2)(a_2^2 + b_2^2)} = \sqrt{a_1^2 a_2^2 + b_1^2 b_2^2 + a_1^2 b_2^2 + a_2^2 b_1^2}$$

which is equal to $|z_1 z_2|$.

Let us show that $|z_1 + z_2| \neq |z_1| + |z_2|$ in general. From part (a) we have

$$z_1 + z_2 = (a_1 + a_2) + i(b_1 + b_2).$$

Hence

$$|z_1 + z_2| = \sqrt{(a_1 + a_2)^2 + (b_1 + b_2)^2}.$$

On the other hand

$$|z_1| + |z_2| = \sqrt{(a_1 + a_2)^2 + (b_1 + b_2)^2}$$

and

$$\sqrt{(a_1 + a_2)^2 + (b_1 + b_2)^2} \neq \sqrt{(a_1 + a_2)^2 + (b_1 + b_2)^2}$$

in general (choose for example $a_1 = 1, b_1 = 0, a_2 = 0, b_2 = 1$).

(d) We want to show that $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$. From part (a) we have

$$z_1 + z_2 = (a_1 + a_2) + i(b_1 + b_2)$$

then

$$\overline{z_1 + z_2} = (a_1 + a_2) - i(b_1 + b_2).$$

On the other hand

$$\overline{z_1} + \overline{z_2} = (a_1 - ib_1) + (a_2 - ib_2) = (a_1 + a_2) - i(b_1 + b_2)$$

which is equal to $\overline{z_1 + z_2}$.

Let us show that $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$. From part (a) we have that

$$z_1 z_2 = (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1).$$

Hence

$$\overline{z_1 z_2} = (a_1 a_2 - b_1 b_2) - i(a_1 b_2 + a_2 b_1).$$

On the other hand,

$$\overline{z_1}\,\overline{z_2} = (a_1 - ib_1)(a_2 - ib_2) = (a_1 a_2 - b_1 b_2) - i(a_1 b_2 + a_2 b_1)$$

which is equal to $\overline{z_1 z_2}$.

4. Establish the following results:
(a)$z + \overline{z} = 2\text{Re}z$,
(b)$z - \overline{z} = 2i\text{Im}z$,
(c)$z_1\overline{z_2} + \overline{z_1}z_2 = 2(z_1\overline{z_2}) = 2\text{Re}(\overline{z_1}z_2)$,
(d)$\text{Re}z \leq |z|$,
(e)$\text{Im}z \leq |z|$,
(f)$|z_1\overline{z_2} + \overline{z_1}z_2| \leq 2|z_1z_2|$,
(g)$(|z_1| - |z_2|)^2 \leq |z_1 + z_2|^2 \leq (|z_1| + |z_2|)^2$. [Use part (f).]

Solution.(a) Let $z = a + ib$, then

$$z + \overline{z} = (a + ib) + (a - ib) = 2a = \text{Re}z;$$

(b) Let $z = a + ib$, then

$$z - \overline{z} = (a + ib) - (a - ib) = 2ib = 2\text{Im}z;$$

(c) Let $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$, then

$$\begin{aligned} z_1\overline{z_2} &= (a_1 + ib_1)(a_2 - ib_2) \\ &= (a_1a_2 + b_1b_2) + i(b_1a_2 - a_1b_2), \end{aligned}$$

$$\begin{aligned} \overline{z_1}z_2 &= (a_1 - ib_1)(a_2 + ib_2) \\ &= (a_1a_2 + b_1b_2) - i(b_1a_2 - a_1b_2) \end{aligned}$$

and

$$z_1\overline{z_2} + \overline{z_1}z_2 = 2(a_1a_2 + b_1b_2).$$

Hence

$$2\text{Re}(z_1\overline{z_2}) = 2\text{Re}(\overline{z_1}z_2) = z_1\overline{z_2} + \overline{z_1}z_2 = 2(a_1a_2 + b_1b_2);$$

(d) Let $z = a + ib$, then

$$|z| = \sqrt{a^2 + b^2} \geq \sqrt{a^2} = |a| \geq a = \text{Re}z;$$

(e) Let $z = a + ib$, then

$$|z| = \sqrt{a^2 + b^2} \geq \sqrt{b^2} = |b| \geq b = \text{Im}z;$$

(f)Let $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$, then using part (c) and part (d) we get,

$$|z_1\overline{z_2} + \overline{z_1}z_2| = 2|\text{Re}(z_1\overline{z_2})| \leq 2|z_1\overline{z_2}|;$$

(g) Let $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$, then

$$\begin{aligned} (|z_1| - |z_2|)^2 &= (|z_1|^2 + |z_2|^2) - 2|z_1||z_2|; \\ |z_1 + z_2|^2 &= (z_1 + z_2)\overline{(z_1 + z_2)} \\ &= (z_1 + z_2)(\overline{z_1} + \overline{z_2}) \\ &= z_1\overline{z_1} + z_1\overline{z_2} + z_2\overline{z_1} + z_2\overline{z_2} \\ &= (|z_1|^2 + |z_2|^2) + (z_1\overline{z_2} + z_2\overline{z_1}); \\ (|z_1| + |z_2|)^2 &= (|z_1|^2 + |z_2|^2) + 2|z_1||z_2|. \end{aligned}$$

To simplify let $A = (|z_1|^2 + |z_2|^2)$. We ant to show that$(|z_1| - |z_2|)^2 \leq |z_1 + z_2|^2 \leq (|z_1| + |z_2|)^2$. The above identities imply that this is equivalent to showing

$$A - 2|z_1 z_2| \leq A + (z_1\overline{z_2} + z_2\overline{z_1}) \leq A + 2|z_1 z_2|.$$

Hence we have to prove that

$$-2|z_1 z_2| \leq (z_1\overline{z_2} + z_2\overline{z_1}) \leq 2|z_1 z_2|.$$

Part (f) implies that

$$|z_1\overline{z_2} + \overline{z_1}z_2| \leq 2|z_1\overline{z_2}|,$$

moreover it is always true that

$$-|z_1\overline{z_2} + \overline{z_1}z_2| \leq (z_1\overline{z_2} + \overline{z_1}z_2) \leq |z_1\overline{z_2} + \overline{z_1}z_2|.$$

Thus we conclude that

$$-2|z_1 z_2| \leq -|z_1\overline{z_2} + \overline{z_1}z_2| \leq (z_1\overline{z_2} + \overline{z_1}z_2) \leq |z_1\overline{z_2} + \overline{z_1}z_2| \leq 2|z_1 z_2|$$

which is what we wanted to show.

5. Express the following quantities in the form $a + ib$, where a and b are real:
(a) $(1+i)^3$, (b) $\frac{1+i}{1-i}$, (c) $e^{\pi i/2}$,
(d) $e^{2+\pi i/4}$, (e) $\sin(\frac{\pi}{4} + 2i)$, (f) $\cosh(2 + \frac{\pi i}{4})$.

Solution.(a) $(1+i)^3 = 1 + 3i + 3i^2 + i^3 = 1 + 3i - 3 - i = -2 + 2i$;
(b) $\frac{1+i}{1-i} = \frac{1+i}{1-i}(\frac{1+i}{1+i}) = \frac{1+2i+i^2}{1-i^2} = \frac{2i}{2} = 0 + i$;
(c) $e^{\pi i/2} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = 0 + i$;
(d)$e^{2=\frac{\pi i}{4}} = e^2(e^{\frac{\pi i}{4}}) = e^2(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}) = e^2(\sqrt{2}/2 + i\sqrt{2}/2) = e^2\sqrt{2}/2 + ie^2\sqrt{2}/2$;
(e)$\sin(\frac{\pi}{4} + 2i) = (\sin\frac{\pi}{4}\cosh 2) + i(\cos\frac{\pi}{4}\sinh 2) = \cosh 2/\sqrt{2} + i\sinh 2/\sqrt{2}$;
(f)$\cosh(2 + \frac{\pi i}{4}) = (\cosh 2\cos\frac{\pi}{4}) + i(\sinh 2\sin\frac{\pi}{4}) = \cosh 2/\sqrt{2} + i\sinh 2/\sqrt{2}$.

9. Prove that e^z possesses no zeros, that the zeros of $\sin z$ and $\cos z$ all lie on the real axis, and that those of $\sinh z$ and $\cosh z$ all lie on the imaginary axis.

Solution. We have to show that $e^z \neq 0$ for all z. Let $z = x + iy$, then

$$e^z = e^{(x+iy)} = e^x e^{iy} = e^x(\cos y + i\sin y).$$

Since $e^x \neq 0$ for all x real, $e^z = 0$ if and only if $\cos y = \sin y = 0$ for a real number y, which is not possible. Hence $e^z \neq 0$ for all z.

Now we want to prove that the zeros of $\sin z$ and $\cos z$ are real. Let $z = x + iy$, then (identity (32) pg 544)

$$\sin(x + iy) = \sin x\cosh y + i(\cos x\sinh y).$$

Therefore $\sin(x + iy) = 0$ if and only if

$$(1)\sin x\cosh y = 0$$

and

$$(2)\cos x \sinh y = 0.$$

Observe that

$$\cosh y = \frac{e^y + e^{-y}}{2} \neq 0,$$

for all y, therefore if (1) holds, we must have $\sin x = 0$ which implies $\cos x \neq 0$. Thus, if (2) holds, we must have

$$\sinh y = \frac{e^y - e^{-y}}{2} = 0$$

which implies $y = 0$. Hence the zeros of $\sin z$ are real.

Analogously, (identity (32) pg 544)

$$\cos(x + iy) = \cos x \cosh y + i(\sin x \sinh y).$$

Therefore if we assume $\cos(x + iy) = 0$, we must have

$$(1)' \cos x \cosh y = 0$$

and

$$(2)' \sin x \sinh y = 0.$$

Since $\cosh y \neq 0$, for all y if (1)' holds, we must have $\cos x = 0$ which implies $\sin x \neq 0$. Thus, if (2)' holds, we must have $\sinh y = 0$ which implies $y = 0$.Hence the zeros of $\cos z$ are real.

Finally we want to show that the zeros of $\sinh z$ and $\cosh z$ are imaginary. The following identity (identity (32)pg 544)

$$\sinh(x + iy) = \sinh x \cos y + i(\cosh x \sin y).$$

implies that $\sinh(x + iy) = 0$ if and only if

$$(3)\sinh x \cos y = 0$$

and

$$(4)\cosh x \sin y = 0.$$

Since $\cosh x \neq 0$, for all x, (4) implies $\sin y = 0$ and so $\cos y \neq 0$. Hence, if (3) holds, we must have $\sinh x = 0$ which implies $x = 0$.Hence the zeros of $\sinh z$ are imaginary. Analogously, (identity (32) pg 544)

$$\cosh(x + iy) = \cosh x \cosh y + i(\sinh x \sin y).$$

Therefore $\cosh(x + iy) = 0$ if and only if

$$(3)' \cosh x \cos y = 0$$

and

$$(4)' \sinh x \sin y = 0.$$

Since $\cosh x \neq 0$, for all x, (3)' implies that $\cos y = 0$ and so $\sin y \neq 0$. Thus, if (4)' holds, we must have $\sinh x = 0$ which implies $x = 0$.Hence the zeros of $\cosh z$ are imaginary.

10.3. **Other Elementary Functions.** .

12. Show that the nth roots of unity are of the form ω_n^k ($k = 0, 1, \ldots, n-1$), where $\omega_n = \cos(2\pi/n) + i\,\sin(2\pi/n)$.

Solution. We need to solve the equation $z^n = 1$. The principal value of the argument θ of unity is $\theta_P = 0$. Hence, from the formula derived in class,

$$z = |1|^{1/n}\, e^{i(\theta_P + 2k\pi)/n} = \left(e^{i2\pi/n}\right)^k = (\omega_n)^k, \quad k = 0, 1, 2, \ldots, n-1,$$

where $\omega_n = e^{i2\pi/n} = \cos(2\pi/n) + i\,\sin(2\pi/n)$.

13. Determine all possible values of the following quantities in the form $a + ib$, and in each case give also the principal value, assuming the definition (39):
(a) $\log(1+i)$, (b) $(i)^{\frac{3}{4}}$, (c) $(1+i)^{\frac{1}{2}}$.

Solution. (a) $\log(1+i) = \log\sqrt{2} + i(2k\pi + \frac{\pi}{4})$, where k is an integer. The principal value is $\log\sqrt{2} + i\frac{\pi}{4}$.

(b) $(i)^{\frac{3}{4}} = \sqrt[4]{(i)^3} = \sqrt[4]{e^{i(\frac{3\pi}{2}+2k\pi)}} = e^{i(\frac{3\pi}{8}+\frac{k\pi}{2})} = \cos(\frac{3\pi}{8} + \frac{k\pi}{2}) + i\sin(\frac{3\pi}{8} + \frac{k\pi}{2})$, where $k = 0, 1, 2, 3$. The principal value is $\cos\frac{3\pi}{8} + i\sin\frac{3\pi}{8}$.

(c) $(1+i)^{\frac{1}{2}} = e^{\frac{1}{2}\log(1+i)} = e^{\frac{1}{2}(\log\sqrt{2}+i(2k\pi+\frac{\pi}{4}))} = \sqrt[4]{2}(\cos(\frac{\pi}{8} + k\pi) + i\sin(\frac{\pi}{8} + k\pi))$, where $k = 0, 1$. The principal value is $\sqrt[4]{2}(\cos\frac{\pi}{8} + i\sin\frac{\pi}{8})$.