18.099/18.06CI - HOMEWORK 2

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Problem 1.

Let S be the space of all homogeneous polynomials of degree p in n variables. Its elements can be expressed as sums of terms of form: $P_m = \prod_{i=1}^{n} \alpha_i^{k_i}, \sum_{i=1}^{n} k_i = p$. In order to solve for the dimension of the space of all such polynomials we need to find the largest possible number of distinct terms. This can be reduced to finding the number of distinct non-negative integer valued solution vectors $\{k_i\}$ to $\sum_{i=1}^{n} k_i = p$ due to the uniqueness of factorization. From elementary combinatorics, the number of positive, $k_i > 0$, solution vectors is $\binom{p-1}{n-1}$. To obtain the number of non-negative solutions, $k_i \geq 0$, we note that the number of such solutions is the same as the number of positive solutions to $\sum_{i=1}^{n} k_i' = p + n$, where $k_i' = k_i + 1$. Hence, we obtain $\binom{p+n-1}{n-1}$ distinct non-negative solution vectors.

Considering the set U of all such distinct non-zero monomial terms we find that they are linearly independent: $\sum c_m P_m = 0,\ P_m \in U,\ P_i \neq P_j,\ i \neq j \Rightarrow$ all $c_m = 0$. Because U is a maximal linearly independent set of elements from S we can consider its elements as a basis for S. The dimension of S is simply the number of elements in its basis and hence dim $S = \binom{p+n-1}{n-1}$. We further note that $\binom{a}{b} > 0$ for $a \geq 0,\ a \geq b$ and thus for the cases $p = 0, n = 1$; $p \leq n; p > n$ our formula works as expected.

Problem 2.

Two finite dimensional linear spaces L and M are isomorphic if and only if for $l = \dim L, m = \dim M,\ l = m$. To prove this let $\{u_i\}_{i=1}^{l}$ be a basis for L and $\{v_j\}_{j=1}^{m}$ be a basis for M. A linear map $f : L \mapsto M$ can be defined by $u_i \mapsto \sum_{j=1}^{m} a_{ij} v_j$. We consider such a map as a system of linear equations, and assert from the properties of matrices that for such map to be invertible, that is for it to be a one-to-one and onto map, it is necessary and sufficient that the matrix of coefficients (a_{ij}) be invertible. This is only possible if $m = l$. Given $m = l$ we can choose I_m as the coefficient matrix and note that it is its own inverse. Thus, L and M are isomorphic if and only if $\dim L = \dim M$.

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