PROFESSOR: Hi everyone. Welcome back. So today, we're going to take a look at first-order linear differential equations with constant coefficients. And specifically, we're going to use integrating factors to solve them. So the equation that we're going to solve today is x dot plus k*x equals 1.

And then in part B we're going to change the right-hand side to e to the minus 5t. And then in part C, we're asked to use the superposition principle to solve x dot plus k*x equals 4 plus 7 e to the minus 5t. So I'll let you think about this for a moment, and I'll come back in a second.

Hi, everyone. Welcome back. So I should mention that every first-order linear differential equation, whether it has constant coefficients or not, can always be solved using an integrating factor. However, in this case, we have a constant coefficient, which is particularly nice. And later on in the course, we're going to learn some even better ways, or quicker ways, to solve linear differential equations with constant coefficients. But for today, we're asked to use an integrating factor.

So for part A, we have the equation x dot plus k*x equals 1. And the first step is to compute the integrating factor. So the integrating factor, which I'll call g of t, it's always going to be an exponential of the integral of the function that appears in front of x.

So in this case, the function is just a constant. It's k. So we have k*dt, which gives us e to the k*t. So once we have the integrating factor, we just multiply our equation through by g of t. And by construction, what the integrating factor does is it lets us combine these two terms on the left-hand side into an exact derivative.

So these two terms are actually the time derivative of the product x times the integrating factor e to the k*t. And then on the right-hand side we just have e to the k*t. So we can just go ahead and integrate both sides of the equation.

And when we do that, the right-hand side becomes the integral of k*t, which is 1 over k e to the k*t plus a constant of integration. And now, just to isolate x, I could divide through by e to the k*t. And I obtain 1 over k plus a constant e to the minus k*t. So here's the solution to the ODE. OK, so this concludes part A.

For part B, we have the equation x dot plus k*x equals e to the negative 5t. So if we take a look at this equation, the only thing that we've changed is the right-hand side. We haven't changed the left-hand side.

And again, if we compute the integrating factor, well, we know that it's the same integrating factor as in part A. And the reason is that the integrating factor only depends on the left-hand side. It only depends on the linear terms. So I can multiply the equation through by the integrating factor again. And when I do this, I'll just combine the terms on the right-hand side.

So this is e to the k*t times e to the minus 5t. And again, by construction, the left-hand side is going to be the same as in part A, the time derivative of x times e to k*t. And now we can go ahead and integrate both sides. OK, so if we integrate this, we end up getting 1 over k minus 5, e to the k minus 5t, plus a constant c.

And if we step back and take a look at this for a second, we see that when k equals 5, we have a problem. Particularly, the denominator vanishes, which would give us 1 over 0. So this equation, this right-hand side, actually only holds when k is not equal to 5. So this is only valid for k not equal to five.

So the question is, what happens when k equals to five? And in this case, we would have x e to the k*t, times the integral of 1 dt, which would just give us t plus a constant c. So in this case, we would have t e to the minus k*t, plus c e to the minus k*t. And this is when k is equal to 5.

Meanwhile, for k not equal to 5, well, we have the solution worked out already. So we can just isolate x, and divide through by e to the k*t. And we have 1 over k minus 5, e to the minus 5t, plus c, e to the minus k*t. And this concludes part B.

So the solution for k equal to 5 is t, e to the minus k*t, which would be e to the minus 5t, plus a constant c times e to the minus 5t. And when k is not equal to 5, we have 1 over k minus 5, e to the minus 5t, plus c e to the minus k*t.

So I'd just like to point out a few things between the solutions for part A and for part B. First off, we note that both part A and part B share a common solution of the form constant c times e to the minus k*t. So this is a term that appears in the solution for both part A and for part B. The reason is this can be thought of as the homogeneous solution to the differential equation. This is the term that solves the differential equation when the right-hand side is set to 0.

Secondly, in part B, if we take a look, when k is not equal to 5, we have the term which is a constant times e to the minus 5t. However, when we have k equal to 5, what happens is we have a term which essentially occurs from forcing the differential equation on resonance, which gives us an extra factor of t times e to the minus 5t. And we'll see more about resonance in the future.

OK, so for part C, we're asked to use superposition. To solve the differential equation x dot plus k*x equals 4 plus 7 e to the minus 5t. Now if we take a look at this differential equation, we already know the solution when the right-hand side is 1 and when the right-hand side is e to the minus 5t. So we've changed the right-hand side now so it's 4 times 1 plus 7 times e to the minus 5t.

So what's the total solution going to be? Well, it's going to be four times our solution when the right-hand side was 1, plus seven times the solution when the right-hand side was e to the minus 5t. This is one of the beautiful things about linear equations. When we add two forcings to the right-hand side, our total solution is just going to be the sum of the solutions to the individual terms.

OK. So what this means is our solution, x, is going to be 4 times the solution when the right-hand side was 1. And in that case, it was 1 over k plus c e to the minus k*t, plus 7 times the solution when the right-hand side was e to the minus 5t. And when k was not equal to 5, this becomes 1 over k minus 5, e to the minus 5t, plus c e to the minus kt.

So if we take a look at the sum of these two terms, I'll denote the two constants as c_1 and c_2. We actually have 4 times c_1 plus 7 times c_2. That's just going to give us a new constant, c_3. So in general, this becomes 4 over k plus 1 over k minus 5, e to the minus 5t, plus a constant c e to the minus k*t. So I can just recombine the 4c e to the minus k*t, and the 7c e to the minus k*t. That together just gives me a new constant, c_3, times e to the minus k*t.

And this is the solution when k is not equal to 5. When k equals to 5, what we do is we just replace this term with t e to the minus k*t. So this is when k is not equal to 5. It should be a 7 here.

So I'll just conclude there. And for summary, we've taken a look at a first-order linear differential equation with a couple different right-hand sides. We've solved them using an integrating factor.

And then what we've done is we've used the superposition principle to solve the same ODE for a right-hand side which is the superposition of multiples of the functions that we've had in part A and for part B. So I'd like to conclude here, and I'll see you next time.