18.03 Recitation 24, May 6, 2010

Matrix exponentials

The equation $\dot{\mathbf{u}} = A\mathbf{u}$ (or the matrix A) is
"stable" if all solutions tend to $\mathbf{0}$ as $t \to \infty$.
"unstable" if most solutions grow without bound as $t \to \infty$.
"neutrally stable" otherwise.

A fundamental matrix for a square matrix A is a matrix of functions, $\Phi(t)$, whose columns are linearly independent solutions to $\dot{\mathbf{u}} = A\mathbf{u}$. The fundamental matrix whose value at $t = 0$ is the identity matrix is the "matrix exponential" e^{At}. It can be computed from any fundamental matrix $\qquad e^{At} = \Phi(t)\Phi(0)^{-1}$.

The solution to $\dot{\mathbf{u}} = A\mathbf{u}$ with initial condition $\mathbf{u}(0)$ is given by $e^{At}\mathbf{u}(0)$.

If $\mathbf{q}$ is constant, and A is invertible, then $\mathbf{u_p}(t) = -A^{-1}\mathbf{q}$ is a solution to the inhomogeneous equation $\dot{\mathbf{u}} = A\mathbf{u} + \mathbf{q}$. The general solution is $\mathbf{u_p} + \mathbf{u_h}$, where $\mathbf{u_h}$ is the general solution of the associated homogeneous equation $\dot{\mathbf{u}} = A\mathbf{u}$.

1. On the trace-determinant plane, where can you guarantee that any matrix with this value of trace and determinant is stable? Unstable? Neutrally stable? Are there there any values of the trace and determinant for which there are matrices exhibiting more than one type of limiting behavior?

2. In this problem, $A = \begin{bmatrix} 1 & 1 \\ -4 & 1 \end{bmatrix}$ and we are interested in the equation $\dot{\mathbf{u}} = A\mathbf{u}$.

(a) Find a fundamental matrix $\Phi(t)$ for A.

(b) Find the exponential matrix e^{At}.

(c) Find the solution to $\dot{\mathbf{u}} = A\mathbf{u}$ with $\mathbf{u}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

(d) Find a solution to $\dot{\mathbf{u}} = A\mathbf{u} + \begin{bmatrix} 5 \\ 10 \end{bmatrix}$. What is the general solution? What is the solution with $\mathbf{u}(0) = \mathbf{0}$?

3. Suppose $\mathbf{u_1}(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (constant function) and $\mathbf{u_2}(t) = \begin{bmatrix} e^t \\ -e^t \end{bmatrix}$ are solutions to the equation $\dot{\mathbf{u}} = B\mathbf{u}$.

(a) What is the general solution? What is the solution $\mathbf{u}(t)$ with $\mathbf{u}(0) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$? What is the solution with $\mathbf{u}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$?

(b) Find a fundamental matrix, and compute the exponential e^{Bt}. What is e^B?

(c) What are the eigenvalues and eigenvectors of B?

(d) What is B?