18.03 Recitation 18, April 13, 2010

Laplace transform

1. Find (from the rules and formulas) the Laplace transform of $u(t)e^{-t}(t^2+1)$.

Here we use the linearity and s-shift rule of the Laplace transform, as well as the formula $\mathcal{L}[e^{-t}] = \frac{1}{s+1}$ and $\mathcal{L}[t^2] = \frac{2}{s^3}$. So the result is given by

$$\begin{aligned}\mathcal{L}[e^{-t}(t^2+1)] &= \mathcal{L}[e^{-t}t^2] + \mathcal{L}[e^{-t}] \\ &= \frac{2}{(s+1)^3} + \frac{1}{s+1} \\ &= \frac{s^2+2s+3}{(s+1)^3}.\end{aligned}$$

2.Let $f(t) = e^{-t}\cos(3t)$. From the rules and tables, what is $F(s) = \mathcal{L}[f(t)]$? Compute the generalized derivative $f'(t)$ and its Laplace transform. Verify the t-derivative rule in this case.

We can read directly from the table that $\mathcal{L}[\cos(3t)] = \frac{s}{s^2+9}$. Hence by the s-shift rule of the Laplace transform,

$$\begin{aligned}F(s) &= \mathcal{L}[e^{-t}\cos(3t)] \\ &= \mathcal{L}[\cos(3t)](s+1) \\ &= \frac{s+1}{(s+1)^2+9} \\ &= \frac{s+1}{s^2+2s+10}.\end{aligned}$$

When $t > 0$, $f'(t) = -e^{-t}(\cos(3t) + 3\sin(3t))$. Since $f(t)$ is actually $f(t)u(t)$, and $f(0+) = 1$, so

$$f'(t) = \delta(t) - u(t)e^{-t}(\cos(3t) + 3\sin(3t)).$$

Again from the table, we know $\mathcal{L}[\sin(3t)] = \frac{3}{s^2+9}$ and $\mathcal{L}[\delta(t)] = 1$. Therefore, by the linearity and the s-shift rule,

$$\begin{aligned}\mathcal{L}[f'(t)] &= \mathcal{L}[\delta(t)] - \mathcal{L}[e^{-t}\cos(3t)] - 3\mathcal{L}[e^{-t}\sin(3t)] \\ &= 1 - \frac{s+1}{s^2+2s+10} - 3\frac{3}{(s+1)^2+9} \\ &= \frac{s^2+s}{s^2+2s+10}.\end{aligned}$$

Hence $\mathcal{L}[f'(t)] = sF(s)$, which verifies the t-derivative rule of the Laplace transform.

3. Find the inverse Laplace transform for each of the following.

$$\frac{2s+1}{s^2+9} \quad , \quad \frac{s^2+2}{s^3-s} \quad , \quad \frac{2}{s^2(s-1)}$$

Using linearity and the cosine and sine formulas, we find that $\mathcal{L}[a\sin(3t)+b\cos(3t)] = \frac{3a+bs}{s^2+9}$. So set $a = 1/3$ and $b = 2$, we have $\mathcal{L}^{-1}[\frac{2s+1}{s^2+9}] = \frac{1}{3}\sin(3t) + 2\cos(3t)$.

For the second one, since the denominator is $s^3 - s = s(s-1)(s+1)$, we expect a combination of 1, e^t, and e^{-t}. For constants a, b and c, we have

$$\begin{aligned}\mathcal{L}[ae^{-t} + b + ce^t] &= \frac{a}{s+1} + \frac{b}{s} + \frac{c}{s-1} \\ &= \frac{a(s^2-s) + b(s^2-1) + c(s^2+s)}{s^3-s} \\ &= \frac{(a+b+c)s^2 + (c-a)s - b}{s^3-s}\end{aligned}$$

So set $b = -2$, and $a = c = 3/2$, $\mathcal{L}^{-1}[\frac{s^2+2}{s^3-s}] = \frac{3}{2}(e^{-t} + e^t) - 2]$.

Similarly for the last one, the denominator is $s^2(s-1)$, so we are looking for constants a, b, c such that

$$\begin{aligned}\frac{2}{s^2(s-1)} &= \mathcal{L}[a + bt + ce^t] \\ &= \frac{a}{s} + \frac{b}{s^2} + \frac{c}{s-1} \\ &= \frac{a(s^2-s) + b(s-1) + cs^2}{s^2(s-1)} \\ &= \frac{(a+c)s^2 + (b-a)s - b}{s^2(s-1)}.\end{aligned}$$

So set $b = -2$, and $a = -c = -2$, then $\mathcal{L}^{-1}[\frac{2}{s^2(s-1)}] = -2(1 + t - e^t)$.

4. Find the unit step and impulse response for the operator $D + 2I$, using the Laplace transform.

We want to find a solution to $\dot{x} + 2x = u(t)$, so we take the Laplace transform of both sides. Denote the Laplace transform of $x(t)$ by $\mathfrak{X}(s)$. By the t-derivative rule and linearity, $\mathcal{L}[\dot{x} + 2x] = s\mathfrak{X}(s) + 2\mathfrak{X}(s)$, and on the right, $\mathcal{L}[u(t)] = \frac{1}{s}$. We find that $\mathfrak{X}(s) = \frac{1}{s(s+2)} = \frac{1/2}{s} + \frac{-1/2}{s+2}$. Taking inverse transforms, we find that the unit step response is $x = \frac{u(t)}{2}(1 - e^{-2t})$.

We do the same thing to the left side of the equation $\dot{x} + 2s = \delta(t)$ as above, but now the Laplace transform of the right side is 1, so $\mathfrak{X}(s) = \frac{1}{s+2}$. We find that unit impulse response is $x = u(t)e^{-2t}$.

5. Solve $\dot{x} + 2x = t^2$ with initial condition $x(0+) = 1$, using Laplace transform.

Since the equation is first order, and the initial condition starts at one, we actually want to take the Laplace transform of both sides of a slightly altered equation: $\dot{x} +$

$2x = t^2 + \delta(t)$. This is because the standard assumption of rest initial conditions requires x to have a jump discontinuity: $x(0^-) = 0$ while $x(0^+) = 1$. This forces $\dot{x}$ to have a $\delta(t)$ term. (An easier way would be to use the t-derivative formula for ordinary derivatives.) We find that $(s+2)\mathfrak{X}(s) = \frac{2}{s^3} + 1$, so we need to find the inverse Laplace transform of $\frac{s^3+2}{s^3(s+2)}$. Using partial fractions, we set

$$
\begin{aligned}
\frac{s^3+2}{s^3(s+2)} &= \frac{a}{s^3} + \frac{b}{s^2} + \frac{c}{s} + \frac{d}{s+2} \\
&= \frac{a(s+2) + b(s^2+2s) + c(s^3+2s^2) + ds^3}{s^3(s+2)} \\
&= \frac{(c+d)s^3 + (b+2c)s^2 + (a+2b)s + 2a}{s^3(s+2)}
\end{aligned}
$$

so $a = 1$, $b = -1/2$, $c = 1/4$, and $d = 3/4$. Taking the inverse Laplace transform, we find that $x = \frac{1}{4}u(t)(2t^2 - 2t + 1 + 3e^{-2t})$.

18.03 Differential Equations
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