18.03 Recitation 18, April 13, 2010

Laplace transform

Rules for the Laplace transform

Definition: $\mathcal{L}[f(t)] = F(s) = \int_{0-}^{\infty} f(t)e^{-st}\,dt$ for $\mathrm{Re}\,(s) \gg 0$.

Linearity: $\mathcal{L}[af(t) + bg(t)] = aF(s) + bG(s)$.

$\mathcal{L}^{-1}$: $F(s)$ essentially determines $f(t)$.

s-shift rule: $\mathcal{L}[e^{rt}f(t)] = F(s-r)$.

t-derivative: $\mathcal{L}[f'(t)] = sF(s)$ where $f'(t)$ denotes the generalized derivative.

If $f(t)$ is continuous for $t > 0$ and $f_r'(t)$ is the ordinary derivative, then

$$\mathcal{L}[f_r'(t)] = sF(s) - f(0+).$$

Formulas for the Laplace transform

$$\mathcal{L}[1] = \frac{1}{s}, \quad \mathcal{L}[\delta(t-a)] = e^{-as}$$
$$\mathcal{L}[e^{rt}] = \frac{1}{s-r}, \quad \mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$$
$$\mathcal{L}[\cos(\omega t)] = \frac{s}{s^2+\omega^2}, \quad \mathcal{L}[\sin(\omega t)] = \frac{\omega}{s^2+\omega^2}$$

1. Find (from the rules and formulas) the Laplace transform of $u(t)e^{-t}(t^2+1)$

2. Let $f(t) = e^{-t}\cos(3t)$. From the rules and tables, what is $F(s) = \mathcal{L}[f(t)]$? Compute the generalized derivative $f'(t)$ and its Laplace transform. Verify the t-derivative rule in this case.

3. Find the inverse Laplace transform for each of the following.

$$\frac{2s+1}{s^2+9}, \quad \frac{s^2+2}{s^3-s}, \quad \frac{2}{s^2(s-1)}$$

4. Find the unit step and impulse response for the operator $D+2I$, using the Laplace transform.

5. Solve $\dot{x} + 2x = t^2$ with initial condition $x(0+) = 1$, using Laplace transform.

18.03 Differential Equations
Spring 2010