Recitation 14, March 30, 2010

Fourier Series

Suppose that $f(t)$ a periodic function and that 2π is a period (so $f(t+2\pi) = f(t)$). (To be honest we also assume that $f(t)$ is piecewise continuous and that $f(a) = \frac{1}{2}(f(a-) + f(a+))$ at points of discontinuity.) Then there is exactly one sequence of numbers $a_0, a_1, \ldots, b_1, \ldots$, for which

$$f(t) = \frac{a_0}{2} + a_1 \cos(t) + a_2 \cos(2t) + \cdots$$

$$+ b_1 \sin(t) + b_2 \sin(2t) + \cdots$$

The Fourier coefficients are defined as the numbers fitting into this expression. They can be calculated using the integral formulas

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(nt)\, dt\,, \qquad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(nt)\, dt$$

but often they can be found more easily than this, starting from some basic examples. One basic example is the standard squarewave: $\mathrm{sq}(t)$ is the odd function of period 2π such that $\mathrm{sq}(t) = 1$ for $0 < t < \pi$.

$$\mathrm{sq}(t) = \frac{4}{\pi}\left(\sin(t) + \frac{\sin(3t)}{3} + \frac{\sin(5t)}{5} + \cdots\right) = \frac{4}{\pi} \sum_{k\,\mathrm{odd}} \frac{\sin(kt)}{k}$$

1. Graph the function $f(t)$ which is even, periodic of period 2π, and such that $f(t) = 2$ for $0 < t < \frac{\pi}{2}$ and $f(t) = 0$ for $\frac{\pi}{2} < t < \pi$. Find its Fourier series in two ways:

(a) Use the integral expressions for the Fourier coefficients. (Is the function even or odd? What can you say right off about the coefficients?)

(b) Express $f(t)$ in terms of $\mathrm{sq}(t)$, substitute the Fourier series for $\mathrm{sq}(t)$, and use some trig id.

(c) Now find the Fourier series for $f(t) - 1$.

2. What is the Fourier series for $\sin^2 t$?

3. Graph the odd function $g(x)$ which is periodic of period π and such that $g(x) = 1$ for $0 < x < \frac{\pi}{2}$. 2π is also a period of $g(x)$, so it has a Fourier series as above. Find it by expressing $g(x)$ in terms of the standard squarewave.

4. Graph the function $h(t)$ which is odd and periodic of period 2π and such that $h(t) = t$ for $0 < t < \frac{\pi}{2}$ and $h(t) = \pi - t$ for $\frac{\pi}{2} < t < \pi$. Find its Fourier series, starting with your solution to **1(c)**.

5. Explain why any function $g(x)$ is a sum of an even function and an odd function in just one way. What is the even part of e^x? What is the odd part?

18.03 Differential Equations
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