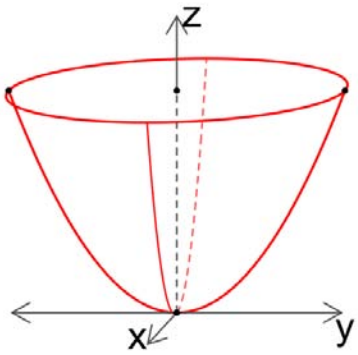
Problems: Flux Through a Paraboloid

Consider the paraboloid $z = x^2 + y^2$. Let S be the portion of this surface that lies below the plane $z = 1$. Let $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + (1 - 2z)\mathbf{k}$.

Calculate the flux of $\mathbf{F}$ across S using the outward normal (the normal pointing away from the z-axis).

Answer: First, draw a picture:



The surface S is a bowl centered on the z-axis. The outward normal $\mathbf{n}$ points away from the outside of the bowl and downward. The region R is the shadow of the bowl – the unit circle in the xy-plane.

We know the z component of $\mathbf{n}$ is negative, so $\mathbf{n}\,dS = \langle z_x, z_y, -1\rangle\,dx\,dy = \langle 2x, 2y, -1\rangle\,dx\,dy$. Thus, $\mathbf{F}\cdot\mathbf{n}\,dS = (2x^2 + 2y^2 + 2z - 1)\,dx\,dy = (4z - 1)\,dx\,dy = (4r^2 - 1)\,dx\,dy$.

$$\begin{aligned}
\iint_S \mathbf{F}\cdot\mathbf{n}\,dS &= \iint_R (4r^2 - 1)\,dx\,dy \\
&= \int_0^{2\pi}\int_0^1 (4r^2 - 1)r\,dr\,d\theta \\
&= \int_0^{2\pi} \frac{1}{2}\,d\theta \\
&= \pi.
\end{aligned}$$

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