V9.1 Surface Integrals

Surface integrals are a natural generalization of line integrals: instead of integrating over a curve, we integrate over a surface in 3-space. Such integrals are important in any of the subjects that deal with continuous media (solids, fluids, gases), as well as subjects that deal with force fields, like electromagnetic or gravitational fields.

Though most of our work will be spent seeing how surface integrals can be calculated and what they are used for, we first want to indicate briefly how they are defined. The surface integral of the (continuous) function $f(x, y, z)$ over the surface S is denoted by

$$\iint_S f(x,y,z)\, dS \ . \tag{1}$$

You can think of dS as the area of an infinitesimal piece of the surface S. To define the integral (1), we subdivide the surface S into small pieces having area ΔS_i, pick a point (x_i, y_i, z_i) in the i-th piece, and form the Riemann sum

$$\sum f(x_i, y_i, z_i) \Delta S_i \ . \tag{2}$$

w As the subdivision of S gets finer and finer, the corresponding sums (2) approach a limit which does not depend on the choice of the points or how the surface was subdivided. The surface integral (1) is defined to be this limit. (The surface has to be smooth and not infinite in extent, and the subdivisions have to be made reasonably, otherwise the limit may not exist, or it may not be unique.)

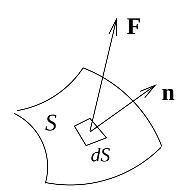
1. The surface integral for flux.

The most important type of surface integral is the one which calculates the flux of a vector field across S. Earlier, we calculated the flux of a plane vector field $\mathbf{F}(x, y)$ across a directed curve in the xy-plane. What we are doing now is the analog of this in space.

We assume that S is *oriented*: this means that S has two sides and one of them has been designated to be the *positive side*. At each point of S there are two unit normal vectors, pointing in opposite directions; the *positively directed* unit normal vector, denoted by $\mathbf{n}$, is the one standing with its base (i.e., tail) on the positive side. If S is a closed surface, like a sphere or cube — that is, a surface with no boundaries, so that it completely encloses a portion of 3-space — then by convention it is oriented so that the outer side is the positive one, i.e., so that $\mathbf{n}$ always points towards the outside of S.

Let $\mathbf{F}(x, y, z)$ be a continuous vector field in space, and S an oriented surface. We define

$$\textsf{\textbf{flux of} } F \textsf{\textbf{ through} } S \;=\; \iint_S (\mathbf{F}\cdot\mathbf{n})\, dS \;=\; \iint_S \mathbf{F}\cdot d\mathbf{S}\ ; \tag{3}$$



the two integrals are the same, but the second is written using the common and suggestive abbreviation $d\mathbf{S} = \mathbf{n}\, dS$.

If $\mathbf{F}$ represents the velocity field for the flow of an incompressible fluid of density 1, then $\mathbf{F}\cdot\mathbf{n}$ represents the component of the velocity in the positive perpendicular direction to the surface, and $\mathbf{F}\cdot\mathbf{n}\, dS$ represents the flow rate across the little infinitesimal piece of surface having area dS. The integral in (3) adds up these flows across the pieces of surface, so that we may interpret (3) as saying

$$\text{(4)} \qquad \textit{flux of } F \textit{ through } S \;=\; \textit{net flow rate across } S,$$

where we count flow in the direction of $\mathbf{n}$ as positive, flow in the opposite direction as negative. More generally, if the fluid has varying density, then the right side of (4) is the net mass transport rate of fluid across S (per unit area, per time unit).

If $\mathbf{F}$ is a force field, then nothing is physically flowing, and one just uses the term "flux" to denote the surface integral, as in (3).