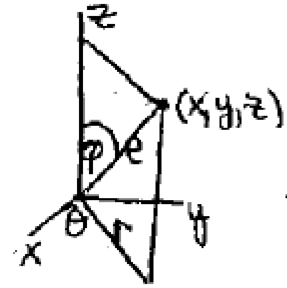
Limits in Spherical Coordinates



Definition of spherical coordinates

$\rho =$ distance to origin, $\quad \rho \geq 0$

$\phi =$ angle to z-axis, $\quad 0 \leq \phi \leq \pi$

$\theta =$ usual $\theta =$ angle of projection to xy-plane with x-axis, $\; 0 \leq \theta \leq 2\pi$

Easy trigonometry gives:

$$\begin{aligned} z &= \rho \cos \phi \\ x &= \rho \sin \phi \cos \theta \\ y &= \rho \sin \phi \sin \theta. \end{aligned}$$

The equations for x and y are most easily deduced by noticing that for r from polar coordinates we have

$$r = \rho \sin \phi.$$

This implies

$$x = r \cos \theta = \rho \sin \phi \cos \theta, \text{ and } y = r \sin \theta = \rho \sin \phi \sin \theta.$$

Going the other way:

$\rho = \sqrt{z^2 + y^2 + z^2} \qquad \phi = \cos^{-1}(z/\rho) \qquad \theta = \tan^{-1}(y/x).$

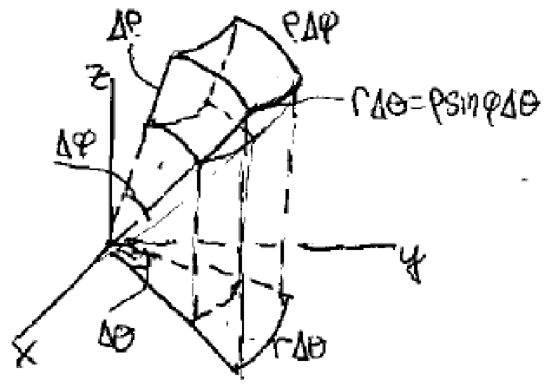
Example: $(x, y, z) = (1, 0, 0) \;\Rightarrow\; \rho = 1,\ \phi = \pi/2,\ \theta = 0$

$(x, y, z) = (0, 1, 0) \;\Rightarrow\; \rho = 1,\ \phi = \pi/2,\ \theta = \pi/2$

$(x, y, z) = (0, 0, 1) \;\Rightarrow\; \rho = 1,\ \phi = 0,\ \theta$ –can be anything

The volume element in spherical coordinates

$$dV = \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta.$$

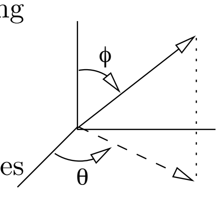


The figure at right shows how we get this. The volume of the curved box is

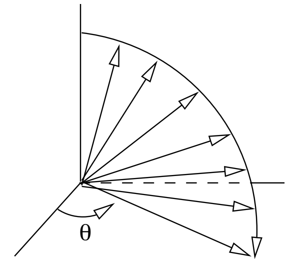
$$\Delta V \approx \Delta\rho \cdot \rho\Delta\phi \cdot \rho \sin \phi \Delta\theta = \rho^2 \sin \phi \, \Delta\rho \, \Delta\phi \, \Delta\theta.$$

Finding limits in spherical coordinates

We use the same procedure as for rectangular and cylindrical coordinates. To calculate the limits for an iterated integral $\int\int\int_D d\rho \, d\phi \, d\theta$ over a region D in 3-space, we are integrating first with respect to ρ. Therefore we

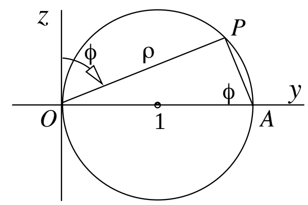


1. Hold ϕ and θ fixed, and let ρ increase. This gives us a ray going out from the origin.
2. Integrate from the ρ-value where the ray enters D to the ρ-value where the ray leaves D. This gives the limits on ρ.



3. Hold θ fixed and let ϕ increase. This gives a family of rays, that form a sort of fan. Integrate over those ϕ-values for which the rays intersect the region D.

4. Finally, supply limits on θ so as to include all of the fans which intersect the region D.



For example, suppose we start with the circle in the yz-plane of radius 1 and center at $(1, 0)$, rotate it about the z-axis, and take D to be that part of the resulting solid lying in the first octant.

First of all, we have to determine the equation of the surface formed by the rotated circle. In the yz-plane, the two coordinates ρ and ϕ are indicated. To see the relation between them when P is on the circle, we see that also angle $OAP = \phi$, since both the angle ϕ and OAP are complements of the same angle, AOP. From the right triangle, this shows the relation is $\rho = 2\sin\phi$.

As the circle is rotated around the z-axis, the relationship stays the same, so $\rho = 2\sin\phi$ is the equation of the whole surface.

To determine the limits of integration, when ϕ and θ are fixed, the corresponding ray enters the region where $\rho = 0$ and leaves where $\rho = 2\sin\phi$.

As ϕ increases, with θ fixed, it is the rays between $\phi = 0$ and $\phi = \pi/2$ that intersect D, since we are only considering the portion of the surface lying in the first octant (and thus above the xy-plane).

Again, since we only want the part in the first octant, we only use θ values from 0 to $\pi/2$. So the iterated integral is

$$\int_0^{\pi/2} \int_0^{\pi/2} \int_0^{2\sin\phi} d\rho\, d\phi\, d\theta.$$

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