

**DAVID JORDAN:** Hello, welcome back to recitation. The problem I'd like to work on with you now is a long one. So it's going to be practice computing line integrals. So to begin with, we have this function of two variables.  $f$  is  $x$  to the fifth plus  $3x y$  cubed. And we have this--  $C$  is the upper semi-circle going from  $(1, 0)$  to  $(-1, 0)$ . So it's this upper semi-circle here that we often consider. And so the first thing that we want to do is to just compute the gradient, capital  $F$ , to be the gradient of this function  $f$ .

And then parts b through d, we're going to compute this line integral of this vector field  $f$  along this curve  $C$ . We're going to compute it in three different ways. So first of all, we're going to compute it directly, just using the definition. And then in Part c, we're going to compute it using the path independence of line integrals and we're going to replace the path  $C$  with a simpler path. And then finally in Part d, we're going to use the fundamental theorem of line integrals.

Now when you do Part b, what I want you to do is set up the integral. You're going to get a very complicated integral that I wouldn't want to try to compute. So just set up the integral completely and then go ahead and move on to Part c and d. So why don't you pause the video and work on that, and we'll check back in a few minutes and we'll solve it together.

Welcome back. I hope you had some luck working these problems. So let's do the easy one first, computing the gradient. So for the gradient we just take the two partial derivatives. So we get  $5x$  to the fourth plus  $3y$  cubed. That's the partial derivative in the  $x$ -direction, and in the  $y$ -direction, we just get  $9xy$  squared.

So now for Part b, we're asked to compute this integral directly. So we have to recall what it means. So first of all, if we go back over here, we have this curve  $C$ . And we need to give a parameterization for it, and so we're going to introduce a parameterization  $r$  of a variable  $t$ , and we're going to use that to do our computations.

So let's set  $r$  of  $t$ -- so this is our usual circle that we're used to working with, so we're just going to take the usual parameterization,  $\cos t$  and  $\sin t$ . And what's important is that the range is going to be from  $t$  equals  $0$  to  $t$  equals  $\pi$ . It's  $t$  equals  $\pi$  because we don't want to go all the way around the circle. We just want to go halfway around until we get to negative  $1$ .

So if that is  $r$  of  $t$ , then we can compute the differential  $dr$  of  $t$ . And so it's going to be just taking the derivative. So we have  $-\sin t$  and  $\cos t$   $dt$ . And so now we can just write out this

line integral directly. So the integral over  $C$  of  $F \cdot dr$  just becomes-- so we have the integral from  $t$  equals 0 to  $\pi$ . Those are the ranges for our curve. And now we're going to take the dot product of  $F$ , which was  $5x$  to the fourth plus  $3y$  cubed,  $9xy$  squared. We're just going to dot this with our  $dr$  vector, which is  $\sin t, \cos t$ . Altogether we have  $dt$ .

And so now, notice that here we've got the variables  $x$  and  $y$ , and here we've got the variables  $t$ , but because of our parameterization, we actually know that, for instance,  $x$  is  $\cos t$  and  $y$  is  $\sin t$ . So we can write this all out. So we have  $5 \cos$  to the fourth  $t$  plus  $3 \sin$  cubed  $t$ . So that's this guy written out in terms of  $t$ . And then we multiply it by a negative sine  $t$ . And then to that we add the other component. So we have plus a  $9$ . So we have  $\cos t$  coming from the  $x$  and another  $\cos t$  here. So we have  $\cos$  squared  $t$ , and we have a sine squared  $t$   $dt$ .

OK, so that's what it means to compute this line integral directly, and it's not something that I look forward to doing. So let's see if we can use path independence to make our lives a little bit simpler. So that's going to be  $c$ . So what we want to do is we want to replace our original curve  $C$  with any other curve that has the same starting point and the same ending point. And the curve that I would like to use is just a straight line connecting them. There's lots of different choices that you could do, but to me this one seems the most natural. So let's give that a try.

So let's let  $r$  of  $t$  be the curve negative  $t, 0$ . Negative  $t$  because we want it to run moving to the left. And then our range is just going to be from minus 1 to 1. So when  $t$  is minus 1, then we get minus the negative 1 and it starts at 1. And when  $t$  is 1 it goes to negative 1. And notice that it goes along the  $y$  equals 0 axis.

So now we can do the same computation that we did before but we can use this curve. So the thing that I want to emphasize is that if we're computing a line integral of a gradient function-- so of a function which is conservative-- then we can use any line and we can use any path that connects the two end points. We can replace our path. And so that's what we did. We replaced  $C_1$  with  $C_2$ . So now this becomes much easier in two ways. So we'll see.

So our range now is just  $t$  goes from minus 1 to 1. And so  $dr$  here is just  $\sin t, 0$ . That's  $dr$ . And there's a  $dt$ . And let's see. So now  $F$ , we had this value for  $F$ , but notice that the  $y$ -coordinate is always 0 along this curve. So the  $y$ -coordinate being 0 means that we just have  $5t$  to the fourth and then 0 here. That's it, because we set  $y$  to be 0 along this curve. So altogether this is a very nice integral to do.

So just taking this dot product, all we have is minus  $5t$  to the fourth  $dt$ . That's simplified

greatly. And we just have minus  $t$  to the fifth from 1 to minus 1. And so we get simply minus 2. So that was a much, much more straightforward integral to do than the one we started with.

Now finally, in  $d$ , we're suggested to use the fundamental theorem of line integrals. So let's remember what that says. That says that if we have any curve and the line integral that we're taking-- if we know that we're taking the line integral not of any vector field, but of a vector field which is already the gradient of  $f$ , then that tells us that this is simply  $f$  of the endpoint minus  $f$  of the starting point of our curve. So really we don't need to do any integral at all.

And so let's see. So recall that  $f$  was  $x$  to the fifth plus  $3x y$  cubed. And so the endpoint-- so we just need to take  $f$  of  $(-1, 0)$  and subtract  $f$  of  $(1, 0)$ . And so plugging this all in together we get minus 1 minus a positive 1. Altogether we get minus 2. And, of course, this does agree with what we did when we computed using the line integrals.