Chain rule with constraints

1. Let $P = (1, 2, 3)$ and assume $f(x, y, z)$ is a differentiable function with $\nabla f = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$ at P. Also assume that x, y and z satisfy the relation $x^3 - y^2 + z = 0$.

Take x and y to be the independent variables and let $g(x, y) = f(x, y, z(x, y))$. Find ∇g at the point (1,2).

Answer: Since f and g are the same, we have $df = dg$. The reason for using two symbols is that f is formally a function of x, y and z and g is formally a function of just x and y.

The gradient gives us the derivatives of f, so at P we have

$$df = dx - 2\,dy + 3\,dz.$$

The constraint gives us

$$3x^2\,dx - 2y\,dy + dz = 0 \;\Rightarrow\; dz = -3x^2\,dx + 2y\,dy.$$

At the point $(1, 2)$ this gives $dz = -3\,dx + 4\,dy$. Substituting this in the equation for df at P gives

$$df = dx - 2\,dy + 3(-3\,dx + 4\,dy) = -8\,dx + 10\,dy.$$

Having written df in terms of dx and dy we have found dg at (1,2). Thus $\dfrac{\partial g}{\partial x} = -8$ and $\dfrac{\partial g}{\partial y} = 10 \;\Rightarrow\; \nabla g = \langle -8, 10\rangle$ at the point (1,2).