2. Partial Differentiation

2A. Functions and Partial Derivatives

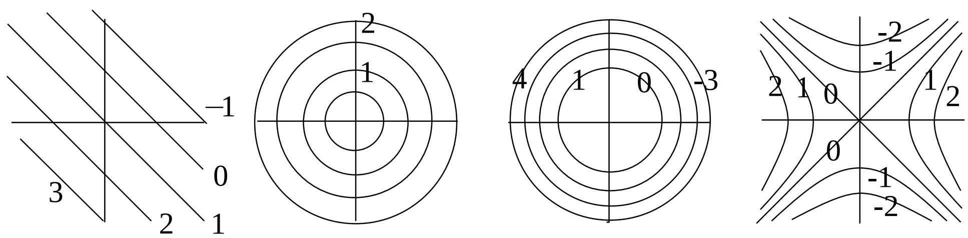
2A-1 In the pictures below, not all of the level curves are labeled. In (c) and (d), the picture is the same, but the labelings are different. In more detail:

b) the origin is the level curve 0; the other two unlabeled level curves are .5 and 1.5;

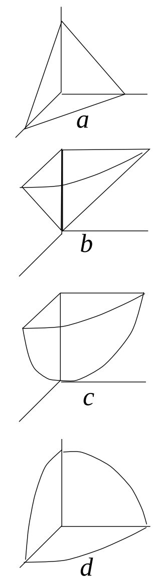
c) on the left, two level curves are labeled; the unlabeled ones are 2 and 3; the origin is the level curve 0;

d) on the right, two level curves are labeled; the unlabeled ones are -1 and -2; the origin is the level curve 1;

The crude sketches of the graph in the first octant are at the right.



a b c, d e



2A-2 a) $f_x = 3x^2y - 3y^2$, $f_y = x^3 - 6xy + 4y$ b) $z_x = \dfrac{1}{y}$, $z_y = -\dfrac{x}{y^2}$

c) $f_x = 3\cos(3x + 2y)$, $f_y = 2\cos(3x + 2y)$

d) $f_x = 2xye^{x^2y}$, $f_y = x^2e^{x^2y}$ e) $z_x = \ln(2x + y) + \dfrac{2x}{2x + y}$, $z_y = \dfrac{x}{2x + y}$

f) $f_x = 2xz$, $f_y = -2z^3$, $f_z = x^2 - 6yz^2$

2A-3 a) both sides are $mnx^{m-1}y^{n-1}$

b) $f_x = \dfrac{y}{(x + y)^2}$, $f_{xy} = (f_x)_y = \dfrac{x - y}{(x + y)^3}$; $f_y = \dfrac{-x}{(x + y)^2}$, $f_{yx} = \dfrac{-(y - x)}{(x + y)^3}$.

c) $f_x = -2x\sin(x^2 + y)$, $f_{xy} = (f_x)_y = -2x\cos(x^2 + y)$;
$f_y = -\sin(x^2 + y)$, $f_{yx} = -\cos(x^2 + y) \cdot 2x$.

d) both sides are $f'(x)g'(y)$.

2A-4 $(f_x)_y = ax + 6y$, $(f_y)_x = 2x + 6y$; therefore $f_{xy} = f_{yx} \Leftrightarrow a = 2$. By inspection, one sees that if $a = 2$, $f(x, y) = x^2y + 3xy^2$ is a function with the given f_x and f_y.

2A-5

a) $w_x = ae^{ax}\sin ay$, $w_{xx} = a^2e^{ax}\sin ay$;
$w_y = e^{ax}a\cos ay$, $w_{yy} = e^{ax}a^2(-\sin ay)$; therefore $w_{yy} = -w_{xx}$.

b) We have $w_x = \dfrac{2x}{x^2 + y^2}$, $w_{xx} = \dfrac{2(y^2 - x^2)}{(x^2 + y^2)^2}$. If we interchange x and y, the function $w = \ln(x^2 + y^2)$ remains the same, while w_{xx} gets turned into w_{yy}; since the interchange just changes the sign of the right hand side, it follows that $w_{yy} = -w_{xx}$.

2B. Tangent Plane; Linear Approximation

2B-1 a) $z_x = y^2$, $z_y = 2xy$; therefore at (1,1,1), we get $z_x = 1$, $z_y = 2$, so that the tangent plane is $z = 1 + (x - 1) + 2(y - 1)$, or $z = x + 2y - 2$.

b) $w_x = -y^2/x^2$, $w_y = 2y/x$; therefore at (1,2,4), we get $w_x = -4$, $w_y = 4$, so that the tangent plane is $w = 4 - 4(x-1) + 4(y-2)$, or $w = -4x + 4y$.

2B-2 a) $z_x = \dfrac{x}{\sqrt{x^2+y^2}} = \dfrac{x}{z}$; by symmetry (interchanging x and y), $z_y = \dfrac{y}{z}$; then the tangent plane is $z = z_0 + \dfrac{x_0}{z_0}(x - x_0) + \dfrac{y_0}{z_0}(y - y_0)$, or $z = \dfrac{x_0}{z_0}x + \dfrac{y_0}{z_0}y$, since $x_0^2 + y_0^2 = z_0^2$.

b) The line is $x = x_0 t$, $y = y_0 t$, $z = z_0 t$; substituting into the equations of the cone and the tangent plane, both are satisfied for all values of t; this shows the line lies on both the cone and tangent plane (this can also be seen geometrically).

2B-3 Letting x, y, z be respectively the lengths of the two legs and the hypotenuse, we have $z = \sqrt{x^2+y^2}$; thus the calculation of partial derivatives is the same as in **2B-2**, and we get $\Delta z \approx \dfrac{3}{5}\Delta x + \dfrac{4}{5}\Delta y$. Taking $\Delta x = \Delta y = .01$, we get $\Delta z \approx \dfrac{7}{5}(.01) = .014$.

2B-4 From the formula, we get $R = \dfrac{R_1 R_2}{R_1 + R_2}$. From this we calculate

$$\frac{\partial R}{\partial R_1} = \left(\frac{R_2}{R_1+R_2}\right)^2, \text{ and by symmetry, } \frac{\partial R}{\partial R_2} = \left(\frac{R_1}{R_1+R_2}\right)^2.$$

Substituting $R_1 = 1$, $R_2 = 2$ the approximation formula then gives $\Delta R = \dfrac{4}{9}\Delta R_1 + \dfrac{1}{9}\Delta R_2$.
By hypothesis, $|\Delta R_i| \le .1$, for $i = 1, 2$, so that $|\Delta R| \le \dfrac{4}{9}(.1) + \dfrac{1}{9}(.1) = \dfrac{5}{9}(.1) \approx .06$; thus

$$R = \frac{2}{3} = .67 \ \pm .06.$$

2B-5 a) We have $f(x,y) = (x+y+2)^2$, $f_x = 2(x+y+2)$, $f_y = 2(x+y+2)$. Therefore

at $(0,0)$, $f_x(0,0) = f_y(0,0) = 4$, $f(0,0) = 4$; linearization is $4 + 4x + 4y$;

at $(1,2)$, $f_x(1,2) = f_y(1,2) = 10$, $f(1,2) = 25$;
linearization is $10(x-1) + 10(y-2) + 25$, or $10x + 10y - 5$.

b) $f = e^x \cos y$; $f_x = e^x \cos y$; $f_y = -e^x \sin y$.

linearization at $(0,0)$: $1 + x$; linearization at $(0, \pi/2)$: $-(y - \pi/2)$

2B-6 We have $V = \pi r^2 h$, $\dfrac{\partial V}{\partial r} = 2\pi r h$, $\dfrac{\partial V}{\partial h} = \pi r^2$; $\Delta V \approx \left(\dfrac{\partial V}{\partial r}\right)_0 \Delta r + \left(\dfrac{\partial V}{\partial h}\right)_0 \Delta h$.
Evaluating the partials at $r = 2$, $h = 3$, we get

$$\Delta V \approx 12\pi \Delta r + 4\pi \Delta h.$$

Assuming the same accuracy $|\Delta r| \le \epsilon$, $|\Delta h| \le \epsilon$ for both measurements, we get

$$|\Delta V| \le 12\pi\,\epsilon + 4\pi\,\epsilon = 16\pi\,\epsilon, \quad \text{which is } < .1 \text{ if } \epsilon < \frac{1}{160\pi} < .002 .$$

2B-7 We have $r = \sqrt{x^2+y^2}$, $\theta = \tan^{-1}\dfrac{y}{x}$; $\dfrac{\partial r}{\partial x} = \dfrac{x}{r}$, $\dfrac{\partial r}{\partial y} = \dfrac{y}{r}$.

Therefore at $(3,4)$, $r = 5$, and $\Delta r \approx \frac{3}{5}\Delta x + \frac{4}{5}\Delta y$. If $|\Delta x|$ and $|\Delta y|$ are both $\le .01$, then

$$|\Delta r| \le \tfrac{3}{5}|\Delta x| + \tfrac{4}{5}|\Delta y| = \tfrac{7}{5}(.01) = .014 \text{ (or } .02).$$

Similarly, $\dfrac{\partial \theta}{\partial x} = \dfrac{-y}{x^2+y^2}$; $\dfrac{\partial \theta}{\partial y} = \dfrac{x}{x^2+y^2}$, so at the point $(3,4)$,

$$|\Delta\theta| \;\le\; |\tfrac{-4}{25}\Delta x| + |\tfrac{3}{25}\Delta y| \;\le\; \tfrac{7}{25}(.01) \;=\; .0028 \text{ (or } .003).$$

Since at $(3,4)$ we have $|r_y| > |r_x|$, r is more sensitive there to changes in y; by analogous reasoning, θ is more sensitive there to x.

2B-9 a) $w = x^2(y+1)$; $w_x = 2x(y+1) = 2$ at $(1,0)$, and $w_y = x^2 = 1$ at $(1,0)$; therefore w is more sensitive to changes in x around this point.

b) To first order approximation, $\Delta w \approx 2\Delta x + \Delta y$, using the above values of the partial derivatives.

If we want $\Delta w = 0$, then by the above, $2\Delta x + \Delta y = 0$, or $\Delta y/\Delta x = -2$.

2C. Differentials; Approximations

2C-1 a) $dw = \dfrac{dx}{x} + \dfrac{dy}{y} + \dfrac{dz}{z}$ b) $dw = 3x^2y^2z\,dx + 2x^3yz\,dy + x^3y^2dz$

c) $dz = \dfrac{2y\,dx - 2x\,dy}{(x+y)^2}$ d) $dw = \dfrac{t\,du - u\,dt}{t\sqrt{t^2-u^2}}$

2C-2 The volume is $V = xyz$; so $dV = yz\,dx + xz\,dy + xy\,dz$. For $x = 5,\ y = 10,\ z = 20$,

$$\Delta V \approx dV = 200\,dx + 100\,dy + 50\,dz,$$

from which we see that $|\Delta V| \le 350(.1)$; therefore $V = 1000 \pm 35$.

2C-3 a) $A = \frac{1}{2}ab\sin\theta$. Therefore, $dA = \frac{1}{2}(b\sin\theta\,da + a\sin\theta\,db + ab\cos\theta\,d\theta)$.

b) $dA = \frac{1}{2}(2\cdot\frac{1}{2}\,da + 1\cdot\frac{1}{2}\,db + 1\cdot 2\cdot\frac{1}{2}\sqrt{3}\,d\theta) = \frac{1}{2}(da + \frac{1}{2}\,db + \sqrt{3}\,d\theta)$;
therefore most sensitive to θ, least senstitive to b, since $d\theta$ and db have respectively the largest and smallest coefficients.

c) $dA = \frac{1}{2}(.02 + .01 + 1.73(.02) \approx \frac{1}{2}(.065) \approx .03$

2C-4 a) $P = \dfrac{kT}{V}$; therefore $dP = \dfrac{k}{V}\,dT - \dfrac{kT}{V^2}\,dV$

b) $V\,dP + P\,dV = k\,dT$; therefore $dP = \dfrac{k\,dT - P\,dV}{V}$.

c) Substituting $P = kT/V$ into (b) turns it into (a).

2C-5 a) $-\dfrac{dw}{w^2} = -\dfrac{dt}{t^2} - \dfrac{du}{u^2} - \dfrac{dv}{v^2}$; therefore $dw = w^2\left(\dfrac{dt}{t^2} + \dfrac{du}{u^2} + \dfrac{dv}{v^2}\right)$.

b) $2u\,du + 4v\,dv + 6w\,dw = 0$; therefore $dw = -\dfrac{u\,du + 2v\,dv}{3w}$.

2D. Gradient; Directional Derivative

2D-1 a) $\nabla f = 3x^2\,\mathbf{i} + 6y^2\,\mathbf{j}$; $(\nabla f)_P = 3\,\mathbf{i} + 6\,\mathbf{j}$; $\left.\dfrac{df}{ds}\right|_{\mathbf{u}} = (3\,\mathbf{i} + 6\,\mathbf{j})\cdot\dfrac{\mathbf{i}-\mathbf{j}}{\sqrt{2}} = -\dfrac{3\sqrt{2}}{2}$

b) $\nabla w = \dfrac{y}{z}\,\mathbf{i} + \dfrac{x}{z}\,\mathbf{j} - \dfrac{xy}{z^2}\,\mathbf{k}$; $(\nabla w)_P = -\,\mathbf{i} + 2\,\mathbf{j} + 2\,\mathbf{k}$; $\left.\dfrac{dw}{ds}\right|_{\mathbf{u}} = (\nabla w)_P\cdot\dfrac{\mathbf{i} + 2\,\mathbf{j} - 2\,\mathbf{k}}{3} = -\dfrac{1}{3}$

c) $\nabla z = (\sin y - y\sin x)\,\mathbf{i} + (x\cos y + \cos x)\,\mathbf{j}$; $(\nabla z)_P = \mathbf{i} + \mathbf{j}$;

$$\left.\frac{dz}{ds}\right|_{\mathbf{u}} = (\,\mathbf{i} + \mathbf{j}\,)\cdot\frac{-3\,\mathbf{i} + 4\,\mathbf{j}}{5} = \frac{1}{5}$$

d) $\nabla w = \dfrac{2\,\mathbf{i} + 3\,\mathbf{j}}{2t+3u}$; $\quad (\nabla w)_P = 2\,\mathbf{i} + 3\,\mathbf{j}$; $\quad \left.\dfrac{dw}{ds}\right|_{\mathbf{u}} = (2\,\mathbf{i} + 3\,\mathbf{j}) \cdot \dfrac{4i - 3\,\mathbf{j}}{5} = -\dfrac{1}{5}$

e) $\nabla f = 2(u+2v+3w)(\,\mathbf{i} + 2\,\mathbf{j} + 3\,\mathbf{k}\,)$; $\quad (\nabla f)_P = 4(\,\mathbf{i} + 2\,\mathbf{j} + 3\,\mathbf{k}\,)$

$$\left.\frac{df}{ds}\right|_{\mathbf{u}} = 4(\,\mathbf{i} + 2\,\mathbf{j} + 3\,\mathbf{k}\,) \cdot \frac{-2\,\mathbf{i} + 2\,\mathbf{j} - \mathbf{k}}{3} = -\frac{4}{3}$$

2D-2 a) $\nabla w = \dfrac{4\,\mathbf{i} - 3\,\mathbf{j}}{4x-3y}$; $\quad (\nabla w)_P = 4\,\mathbf{i} - 3\,\mathbf{j}$

$\left.\dfrac{dw}{ds}\right|_{\mathbf{u}} = (4\,\mathbf{i} - 3\,\mathbf{j}\,) \cdot \mathbf{u}$ has maximum 5, in the direction $\mathbf{u} = \dfrac{4\,\mathbf{i} - 3\,\mathbf{j}}{5}$,
and minimum -5 in the opposite direction.

$\left.\dfrac{dw}{ds}\right|_{\mathbf{u}} = 0$ in the directions $\pm\dfrac{3\,\mathbf{i} + 4\,\mathbf{j}}{5}$.

b) $\nabla w = \langle y+z, x+z, x+y\rangle$; $\quad (\nabla w)_P = \langle 1, 3, 0\rangle$;

max $\left.\dfrac{dw}{ds}\right|_{\mathbf{u}} = \sqrt{10}$, direction $\dfrac{\mathbf{i} + 3\,\mathbf{j}}{\sqrt{10}}$; $\quad$ min $\left.\dfrac{dw}{ds}\right|_{\mathbf{u}} = -\sqrt{10}$, direction $-\dfrac{\mathbf{i} + 3\,\mathbf{j}}{\sqrt{10}}$;

$\left.\dfrac{dw}{ds}\right|_{\mathbf{u}} = 0$ in the directions $\mathbf{u} = \pm\dfrac{-3\,\mathbf{i} + \mathbf{j} + c\,\mathbf{k}}{\sqrt{10+c^2}}$ (for all c)

c) $\nabla z = 2\sin(t-u)\cos(t-u)(\,\mathbf{i} - \mathbf{j}\,) = \sin 2(t-u)(\,\mathbf{i} - \mathbf{j}\,)$; $\quad (\nabla z)_P = \mathbf{i} - \mathbf{j}$;

max $\left.\dfrac{dz}{ds}\right|_{\mathbf{u}} = \sqrt{2}$, direction $\dfrac{\mathbf{i} - \mathbf{j}}{\sqrt{2}}$; $\quad$ min $\left.\dfrac{dz}{ds}\right|_{\mathbf{u}} = -\sqrt{2}$, direction $-\dfrac{-\,\mathbf{i} + \mathbf{j}}{\sqrt{2}}$;

$\left.\dfrac{dz}{ds}\right|_{\mathbf{u}} = 0$ in the directions $\pm\dfrac{\mathbf{i} + \mathbf{j}}{\sqrt{2}}$

2D-3 a) $\nabla f = \langle y^2z^3, 2xyz^3, 3xy^2z^2\rangle$; $\quad (\nabla f)_P = \langle 4, 12, 36\rangle$; $\quad$ normal at P: $\langle 1, 3, 9\rangle$;
tangent plane at P: $x + 3y + 9z = 18$

b) $\nabla f = \langle 2x, 8y, 18z\rangle$; $\quad$ normal at P: $\langle 1, 4, 9\rangle$, $\quad$ tangent plane: $x + 4y + 9z = 14$.

c) $(\nabla w)_P = \langle 2x_0, 2y_0, -2z_0\rangle$; $\quad$ tangent plane: $x_0(x-x_0) + y_0(y-y_0) - z_0(z-z_0) = 0$,
or $x_0x + y_0y - z_0z = 0$, since $x_0^2 + y_0^2 - z_0^2 = 0$.

2D-4 a) $\nabla T = \dfrac{2x\,\mathbf{i} + 2y\,\mathbf{j}}{x^2+y^2}$; $\quad (\nabla T)_P = \dfrac{2\,\mathbf{i} + 4\,\mathbf{j}}{5}$;

T is increasing at P most rapidly in the direction of $(\nabla T)_P$, which is $\dfrac{\mathbf{i} + 2\,\mathbf{j}}{\sqrt{5}}$.

b) $|\nabla T| = \dfrac{2}{\sqrt{5}}$ = rate of increase in direction $\dfrac{\mathbf{i} + 2\,\mathbf{j}}{\sqrt{5}}$. Call the distance to go Δs, then

$$\frac{2}{\sqrt{5}}\,\Delta s = .20 \quad \Rightarrow \quad \Delta s = \frac{.2\sqrt{5}}{2} = \frac{\sqrt{5}}{10} \approx .22.$$

c) $\left.\dfrac{dT}{ds}\right|_{\mathbf{u}} = (\nabla T)_P \cdot \mathbf{u} = \dfrac{2\,\mathbf{i} + 4\,\mathbf{j}}{5} \cdot \dfrac{\mathbf{i} + \mathbf{j}}{\sqrt{2}} = \dfrac{6}{5\sqrt{2}}$;

$$\frac{6}{5\sqrt{2}}\,\Delta s = .12 \quad \Rightarrow \quad \Delta s = \frac{5\sqrt{2}}{6}\,(.12) \;\approx\; (.10)(\sqrt{2}) \;\approx\; .14$$

d) In the directions orthogonal to the gradient: $\pm\dfrac{2\,\mathbf{i} - \mathbf{j}}{\sqrt{5}}$

2D-5 a) isotherms = the level surfaces $x^2 + 2y^2 + 2z^2 = c$, which are ellipsoids.

b) $\nabla T = \langle 2x, 4y, 4z\rangle$; $(\nabla T)_P = \langle 2,4,4\rangle$; $|(\nabla T)_P| = 6$;
for most rapid decrease, use direction of $-(\nabla T)_P$: $-\frac{1}{3}\langle 1,2,2\rangle$

c) let Δs be distance to go; then $-6(\Delta s) = -1.2$; $\Delta s \approx .2$

d) $\left.\dfrac{dT}{ds}\right|_{\mathbf{u}} = (\nabla T)_P\cdot\mathbf{u} = \langle 2,4,4\rangle \cdot \dfrac{\langle 1,-2,2\rangle}{3} = \dfrac{2}{3}$; $\quad \dfrac{2}{3}\Delta s \approx .10 \;\Rightarrow\; \Delta s \approx .15.$

2D-6 $\nabla uv = \langle (uv)_x, (uv)_y\rangle = \langle uv_x + vu_x, uv_y + vu_y\rangle = \langle uv_x, uv_y\rangle + \langle vu_x + vu_y\rangle = u\nabla v + v\nabla u$

$$\nabla(uv) = u\nabla v + v\nabla u \quad\Rightarrow\quad \nabla(uv)\cdot\mathbf{u} = u\nabla v\cdot\mathbf{u} + v\nabla u\cdot\mathbf{u} \quad\Rightarrow\quad \left.\frac{d(uv)}{ds}\right|_{\mathbf{u}} = u\left.\frac{dv}{ds}\right|_{\mathbf{u}} + v\left.\frac{du}{ds}\right|_{\mathbf{u}}.$$

2D-7 At P, let $\nabla w = a\,\mathbf{i} + b\,\mathbf{j}$. Then

$$a\,\mathbf{i} + b\,\mathbf{j}\cdot\frac{\mathbf{i}+\mathbf{j}}{\sqrt{2}} = 2 \quad\Rightarrow\quad a + b = 2\sqrt{2}$$
$$a\,\mathbf{i} + b\,\mathbf{j}\cdot\frac{\mathbf{i}-\mathbf{j}}{\sqrt{2}} = 1 \quad\Rightarrow\quad a - b = \sqrt{2}$$

Adding and subtracting the equations on the right, we get $a = \frac{3}{2}\sqrt{2}$, $b = \frac{1}{2}\sqrt{2}$.

2D-8 We have $P(0,0,0) = 32$; we wish to decrease it to 31.1 by traveling the shortest distance from the origin $\mathbf{0}$; for this we should travel in the direction of $-(\nabla P)_{\mathbf{0}}$.

$$\nabla P = \langle (y+2)e^z, (x+1)e^z, (x+1)(y+2)e^z\rangle; \quad (\nabla P)_{\mathbf{0}} = \langle 2,1,2\rangle. \qquad |(\nabla P)_{\mathbf{0}}| = 3.$$

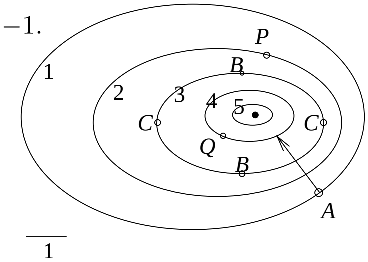
Since $(-3)\cdot(\Delta s) = -.9 \quad\Rightarrow\quad \Delta s = .3$, we should travel a distance .3 in the direction of $-(\nabla P)_{\mathbf{0}}$. Since $|-\langle 2,1,2\rangle| = 3$, the distance .3 will be $\frac{1}{10}$ of the distance from $(0,0,0)$ to $(-2,-1,-2)$, which will bring us to $(-.2,-.1,-.2)$.

2D-9 In these, we use $\left.\dfrac{dw}{ds}\right|_{\mathbf{u}} \approx \dfrac{\Delta w}{\Delta s}$: we travel in the direction $\mathbf{u}$ from a given point P to the nearest level curve C; then Δs is the distance traveled (estimate it by using the unit distance), and Δw is the corresponding change in w (estimate it by using the labels on the level curves).

a) The *direction* of ∇f is perpendicular to the level curve at A, in the increasing sense (the "uphill" direction). The *magnitude* of ∇f is the directional derivative in that direction: from the picture, $\dfrac{\Delta w}{\Delta s} \approx \dfrac{1}{.5} = 2$.

b), c) $\dfrac{\partial w}{\partial x} = \left.\dfrac{dw}{ds}\right|_{\mathbf{i}}$, $\dfrac{\partial w}{\partial y} = \left.\dfrac{dw}{ds}\right|_{\mathbf{j}}$, so B will be where $\mathbf{i}$ is tangent to the level curve and C where $\mathbf{j}$ is tangent to the level curve.

d) At P, $\dfrac{\partial w}{\partial x} = \left.\dfrac{dw}{ds}\right|_{\mathbf{i}} \approx \dfrac{\Delta w}{\Delta s} \approx \dfrac{-1}{5/3} = -.6$; $\quad \dfrac{\partial w}{\partial y} = \left.\dfrac{dw}{ds}\right|_{\mathbf{j}} \approx \dfrac{\Delta w}{\Delta s} \approx \dfrac{-1}{1} = -1.$



e) If $\mathbf{u}$ is the direction of $\mathbf{i} + \mathbf{j}$, we have $\left.\dfrac{dw}{ds}\right|_{u} \approx \dfrac{\Delta w}{\Delta s} \approx \dfrac{1}{.5} = 2$

f) If $\mathbf{u}$ is the direction of $\mathbf{i} - \mathbf{j}$, we have $\left.\dfrac{dw}{ds}\right|_{u} \approx \dfrac{\Delta w}{\Delta s} \approx \dfrac{-1}{5/4} = -.8$

g) The gradient is 0 at a local extremum point: here at the point marked giving the location of the hilltop.

2E. Chain Rule

2E-1

a) (i) $\dfrac{dw}{dt} = \dfrac{\partial w}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial w}{\partial y}\dfrac{dy}{dt} + \dfrac{\partial w}{\partial z}\dfrac{dz}{dt} = yz \cdot 1 + xz \cdot 2t + xy \cdot 3t^2 = t^5 + 2t^5 + 3t^5 = 6t^5$

(ii) $w = xyz = t^6$; $\quad \dfrac{dw}{dt} = 6t^5$

b) (i) $\dfrac{dw}{dt} = \dfrac{\partial w}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial w}{\partial y}\dfrac{dy}{dt} = 2x(-\sin t) - 2y(\cos t) = -4\sin t\cos t$

(ii) $w = x^2 - y^2 = \cos^2 t - \sin^2 t = \cos 2t$; $\quad \dfrac{dw}{dt} = -2\sin 2t$

c) (i) $\dfrac{dw}{dt} = \dfrac{2u}{u^2+v^2}(-2\sin t) + \dfrac{2v}{u^2+v^2}(2\cos t) = -\cos t\sin t + \sin t\cos t = 0$

(ii) $w = \ln(u^2+v^2) = \ln(4\cos^2 t + 4\sin^2 t) = \ln 4$; $\quad \dfrac{dw}{dt} = 0.$

2E-2 a) The value $t = 0$ corresponds to the point $(x(0), y(0)) = (1, 0) = P$.

$$\left.\frac{dw}{dt}\right|_0 = \left.\frac{\partial w}{\partial x}\right|_P \left.\frac{dx}{dt}\right|_0 + \left.\frac{\partial w}{\partial y}\right|_P \left.\frac{dy}{dt}\right|_0 = -2\,\sin t + 3\,\cos t\Big]_0 \;=\; 3.$$

b) $\dfrac{dw}{dt} = \dfrac{\partial w}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial w}{\partial y}\dfrac{dy}{dt} = y(-\sin t) + x(\cos t) = -\sin^2 t + \cos^2 t = \cos 2t.$

$\dfrac{dw}{dt} = 0$ when $2t = \dfrac{\pi}{2} + n\pi$, therefore when $t = \dfrac{\pi}{4} + \dfrac{n\pi}{2}$.

c) $t = 1$ corresponds to the point $(x(1), y(1), z(1)) = (1, 1, 1)$.

$$\left.\frac{df}{dt}\right|_1 = 1 \cdot \left.\frac{dx}{dt}\right|_1 - 1 \cdot \left.\frac{dy}{dt}\right|_1 + 2 \cdot \left.\frac{dz}{dt}\right|_1 = 1 \cdot 1 - 1 \cdot 2 + 2 \cdot 3 = 5.$$

d) $\dfrac{df}{dt} = 3x^2y\dfrac{dx}{dt} + (x^3+z)\dfrac{dy}{dt} + y\dfrac{dz}{dt} = 3t^4 \cdot 1 + 2x^3 \cdot 2t + t^2 \cdot 3t^2 = 10t^4.$

2E-3 a) Let $w = uv$, where $u = u(t),\ v = v(t)$; $\quad \dfrac{dw}{dt} = \dfrac{\partial w}{\partial u}\dfrac{du}{dt} + \dfrac{\partial w}{\partial v}\dfrac{dv}{dt} = v\,\dfrac{du}{dt} + u\,\dfrac{dv}{dt}.$

b) $\dfrac{d(uvw)}{dt} = vw\dfrac{du}{dt} + uw\dfrac{dv}{dt} + uv\dfrac{dw}{dt}$; $\quad e^{2t}\sin t + 2te^{2t}\sin t + te^{2t}\cos t$

2E-4 The values $u = 1,\ v = 1$ correspond to the point $x = 0,\ y = 1$. At this point,

$$\frac{\partial w}{\partial u} \;=\; \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} \;=\; 2 \cdot 2u + 3 \cdot v \;=\; 2 \cdot 2 + 3 \;=\; 7.$$

$$\frac{\partial w}{\partial v} \;=\; \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} \;=\; 2 \cdot (-2v) + 3 \cdot u \;=\; 2 \cdot (-2) + 3 \cdot 1 \;=\; -1.$$

2E-5 a) $w_r = w_x x_r + w_y y_r = w_x\cos\theta + w_y\sin\theta$

$w_\theta = w_x x_\theta + w_y y_\theta = w_x(-r\sin\theta) + w_y(r\cos\theta)$

Therefore,

$(w_r)^2 + (w_\theta/r)^2$

$= (w_x)^2(\cos^2\theta + \sin^2\theta) + (w_y)^2(\sin^2\theta + \cos^2\theta) + 2w_xw_y\cos\theta\sin\theta - 2w_xw_y\sin\theta\cos\theta$

$= (w_x)^2 + (w_y)^2.$

b) The point $r = \sqrt{2}$, $\theta = \pi/4$ in polar coordinates corresponds in rectangular coordinates to the point $x = 1,\ y = 1$. Using the chain rule equations in part (a),

$$w_r = w_x \cos\theta + w_y \sin\theta; \qquad w_\theta = w_x(-r\sin\theta) + w_y(r\cos\theta)$$

but evaluating all the partial derivatives at the point, we get

$$w_r = 2\cdot\tfrac{1}{2}\sqrt{2} - 1\cdot\tfrac{1}{2}\sqrt{2} = \tfrac{1}{2}\sqrt{2}; \quad \frac{w_\theta}{r} = 2(-\tfrac{1}{2})\sqrt{2} - \tfrac{1}{2}\sqrt{2} = -\tfrac{3}{2}\sqrt{2};$$

$$(w_r)^2 + \frac{1}{r}(w_\theta)^2 = \tfrac{1}{2} + \tfrac{9}{2} = 5; \qquad (w_x)^2 + (w_y)^2 = 2^2 + (-1)^2 = 5.$$

2E-6 $\quad w_u = w_x\cdot 2u + w_y \cdot 2v; \quad w_v = w_x\cdot(-2v) + w_y\cdot 2u$, by the chain rule.
Therefore

$$\begin{aligned}(w_u)^2 + (w_v)^2 &= [4u^2(w_x) + 4v^2(w_y)^2 + 4uvw_xw_y] + [4v^2(w_x) + 4u^2(w_y)^2 - 4uvw_xw_y] \\ &= 4(u^2+v^2)[(w_x)^2 + (w_y)^2].\end{aligned}$$

2E-7 By the chain rule, $f_u = f_x x_u + f_y y_u, \quad f_v = f_x x_v + f_y y_v;$ therefore

$$\langle f_u\ f_v\rangle = \langle f_x\ f_y\rangle \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix}$$

2E-8 a) By the chain rule for functions of one variable,

$$\frac{\partial w}{\partial x} = f'(u)\cdot\frac{\partial u}{\partial x} = f'(u)\cdot -\frac{y}{x^2}; \qquad \frac{\partial w}{\partial y} = f'(u)\cdot\frac{\partial u}{\partial y} = f'(u)\cdot\frac{1}{x};$$

Therefore,

$$x\,\frac{\partial w}{\partial x} + y\,\frac{\partial w}{\partial y} = f'(u)\cdot -\frac{y}{x} + f'(u)\cdot\frac{y}{x} = 0.$$

2F. Maximum-minimum Problems

2F-1 In these, denote by $D = x^2+y^2+z^2$ the square of the distance from the point (x,y,z) to the origin; then the point which minimizes D will also minimize the actual distance.

a) Since $z^2 = \dfrac{1}{xy}$, we get on substituting, $D = x^2 + y^2 + \dfrac{1}{xy}$. with x and y independent; setting the partial derivatives equal to zero, we get

$$D_x = 2x - \frac{1}{x^2y} = 0; \quad D_y = 2y - \frac{1}{y^2x} = 0; \quad \text{or} \quad 2x^2 = \frac{1}{xy}, \quad 2y^2 = \frac{1}{xy}.$$

Solving, we see first that $x^2 = \dfrac{1}{2xy} = y^2$, from which $y = \pm x$.

If $y = x$, then $x^4 = \frac{1}{2}$ and $x = y = 2^{-1/4}$, and so $z = 2^{1/4}$; if $y = -x$, then $x^4 = -\frac{1}{2}$ and there are no solutions. Thus the unique point is $(1/2^{1/4}, 1/2^{1/4}, 2^{1/4})$.

b) Using the relation $x^2 = 1 + yz$ to eliminate x, we have $D = 1 + yz + y^2 + z^2$, with y and z independent; setting the partial derivatives equal to zero, we get

$$D_y = 2y + z = 0, \quad D_z = 2z + y = 0;$$

solving, these equations only have the solution $y = z = 0$; therefore $x = \pm 1$, and there are two points: $(\pm 1, 0, 0)$, both at distance 1 from the origin.

2F-2 Letting x be the length of the ends, y the length of the sides, and z the height, we have

$$\text{total area of cardboard} \quad A = 3xy + 4xz + 2yz, \qquad \text{volume} \quad V = xyz = 1.$$

Eliminating z to make the remaining variables independent, and equating the partials to zero, we get

$$A = 3xy + \frac{4}{y} + \frac{2}{x}; \qquad A_x = 3y - \frac{2}{x^2} = 0, \quad A_y = 3x - \frac{4}{y^2} = 0.$$

From these last two equations, we get

$$3xy = \frac{2}{x}, \quad 3xy = \frac{4}{y} \quad \Rightarrow \quad \frac{2}{x} = \frac{4}{y} \quad \Rightarrow \quad y = 2x$$

$$\Rightarrow \quad 3x^3 = 1 \quad \Rightarrow \quad x = \frac{1}{3^{1/3}}, \;\; y = \frac{2}{3^{1/3}}, \;\; z = \frac{1}{xy} = \frac{3^{2/3}}{2} = \frac{3}{2 \cdot 3^{1/3}};$$

therefore the proportions of the most economical box are $x : y : z = 1 : 2 : \frac{3}{2}$.

2F-5 The cost is $C = xy + xz + 4yz + 4xz$, where the successive terms represent in turn the bottom, back, two sides, and front; i.e., the problem is:

$$\text{minimize:} \quad C = xy + 5xz + 4yz, \qquad \text{with the constraint:} \quad xyz = V = 2.5$$

Substituting $z = V/xy$ into C, we get

$$C = xy + \frac{5V}{y} + \frac{4V}{x}; \qquad \frac{\partial C}{\partial x} = y - \frac{4V}{x^2}, \quad \frac{\partial C}{\partial y} = x - \frac{5V}{y^2}.$$

We set the two partial derivatives equal to zero and solving the resulting equations simultaneously, by eliminating y; we get $x^3 = \dfrac{16V}{5} = 8$, (using $V = 5/2$), so $x = 2$, $y = \frac{5}{2}$, $z = \frac{1}{2}$.

2G. Least-squares Interpolation

2G-1 Find $y = mx + b$ that best fits $(1,1)$, $(2,3)$, $(3,2)$.

$$\begin{aligned} D &= (m+b-1)^2 + (2m+b-3)^2 + (3m+b-2)^2 \\ \frac{\partial D}{\partial m} &= 2(m+b-1) + 4(2m+b-3) + 6(3m+b-2) \;=\; 2(14m+6b-13) \\ \frac{\partial D}{\partial b} &= 2(m+b-1) + 2(2m+b-3) + 2(3m+b-2) \;=\; 2(6m+3b-6). \end{aligned}$$

Thus the equations $\dfrac{\partial D}{\partial m} = 0$ and $\dfrac{\partial D}{\partial b} = 0$ are $\begin{cases} 14m + 6b = 13 \\ 6m + 3b = 6 \end{cases}$, whose solution is $m = \frac{1}{2}$, $b = 1$, and the line is $y = \frac{1}{2}x + 1$.

2G-4 $D = \sum_i (a + bx_i + cy_i - z_i)^2$. The equations are

$$\begin{aligned} \partial D/\partial a &= \textstyle\sum 2(a + bx_i + cy_i - z_i) \;=\; 0 \\ \partial D/\partial b &= \textstyle\sum 2x_i(a + bx_i + cy_i - z_i) \;=\; 0 \\ \partial D/\partial c &= \textstyle\sum 2y_i(a + bx_i + cy_i - z_i) \;=\; 0 \end{aligned}$$

Cancel the 2's; the equations become (on the right, $\mathbf{x} = [x_1, \ldots, x_n]$, $\mathbf{1} = [1, \ldots, 1]$, etc.)

$$\begin{aligned} na \;+\; (\textstyle\sum x_i)b \;+\; (\sum y_i)c &= \textstyle\sum z_i & \qquad\qquad n\,a + (\mathbf{x}\cdot\mathbf{1})\,b + (\mathbf{y}\cdot\mathbf{1})\,c &= \mathbf{z}\cdot\mathbf{1} \\ (\textstyle\sum x_i)a + (\sum x_i^2)b + (\sum x_i y_i)c &= \textstyle\sum x_i z_i & \text{or} \qquad (\mathbf{x}\cdot\mathbf{1})\,a + (\mathbf{x}\cdot\mathbf{x})\,b + (\mathbf{x}\cdot\mathbf{y})\,c &= \mathbf{x}\cdot\mathbf{z} \\ (\textstyle\sum y_i)a + (\sum x_i y_i)b + (\sum y_i^2)c &= \textstyle\sum y_i z_i & (\mathbf{y}\cdot\mathbf{1})\,a + (\mathbf{x}\cdot\mathbf{y}\,)b + (\mathbf{y}\cdot\mathbf{y})\,c &= \mathbf{y}\cdot\mathbf{z} \end{aligned}$$

2H. Max-min: 2nd Derivative Criterion; Boundary Curves

2H-1

a) $f_x = 0: \ 2x - y = 3; \quad f_y = 0: \ -x - 4y = 3 \qquad$ critical point: $(1, -1)$

$A = f_{xx} = 2;\ B = f_{xy} = -1;\ C = f_{yy} = -4; \qquad AC - B^2 = -9 < 0$; saddle point

b) $f_x = 0: \ 6x + y = 1; \quad f_y = 0: \ x + 2y = 2 \qquad$ critical point: $(0, 1)$

$A = f_{xx} = 6;\ B = f_{xy} = 1;\ C = f_{yy} = 2; \qquad AC - B^2 = 11 > 0$; local minimum

c) $f_x = 0: \ 8x^3 - y = 0; \quad f_y = 0: \ 2y - x = 0; \qquad$ eliminating y, we get

$16x^3 - x = 0$, or $x(16x^2 - 1) = 0 \ \Rightarrow\ x = 0,\ x = \frac{1}{4},\ x = -\frac{1}{4}$, giving the critical points

$$(0,0), \quad (\tfrac{1}{4}, \tfrac{1}{8}), \quad (-\tfrac{1}{4}, -\tfrac{1}{8}).$$

Since $f_{xx} = 24x^2, \quad f_{xy} = -1, \quad f_{yy} = 2$, we get for the three points respectively:

$(0,0): \ \Delta = -1$ (saddle); $\quad (\frac{1}{4}, \frac{1}{8}): \ \Delta = 2$ (minimum); $\qquad (-\frac{1}{4}, -\frac{1}{8}): \ \Delta = 2$ (minimum)

d) $f_x = 0: \quad 3x^2 - 3y = 0; \qquad f_y = 0: \quad -3x + 3y^2 = 0$. Eliminating y gives

$-x + x^4 = 0$, or $x(x^3 - 1) = 0 \quad \Rightarrow \quad x = 0,\ y = 0$ or $x = 1,\ y = 1$.

Since $f_{xx} = 6x, \quad f_{xy} = -3, \quad f_{yy} = 6y$, we get for the two critical points respectively:

$(0,0): \quad AC - B^2 = -9$ (saddle); $\qquad (1,1): \quad AC - B^2 = 27$ (minimum)

e) $f_x = 0: \quad 3x^2(y^3 + 1) = 0; \qquad f_y = 0: \quad 3y^2(x^3 + 1) = 0$; solving simultaneously, we get from the first equation that either $x = 0$ or $y = -1$; finding in each case the other coordinate then leads to the two critical points $(0,0)$ and $(-1,-1)$.

Since $f_{xx} = 6x(y^3 + 1), \quad f_{xy} = 3x^2 \cdot 3y^2, \quad f_{yy} = 6y(x^3 + 1)$, we have

$(-1,-1): \quad AC - B^2 = -9$ (saddle); $\qquad (0,0): \quad AC - B^2 = 0$, test fails.

(By studying the behavior of $f(x,y)$ on the lines $y = mx$, for different values of m, it is possible to see that also $(0,0)$ is a saddle point.)

2H-3 The region R has no critical points; namely, the equations $f_x = 0$ and $f_y = 0$ are

$$2x + 2 = 0, \quad 2y + 4 = 0 \quad \Rightarrow \quad x = -1,\ y = -2,$$

but this point is not in R. We therefore investigate the diagonal boundary of R, using the parametrization $x = t,\ y = -t$. Restricted to this line, $f(x,y)$ becomes a function of t alone, which we denote by $g(t)$, and we look for its maxima and minima.

$$g(t) = f(t, -t) = 2t^2 - 4t - 1; \qquad g'(t) = 4t - 2, \text{ which is 0 at } t = 1/2.$$

This point is evidently a minimum for $g(t)$; there is no maximum: $g(t)$ tends to ∞. Therefore for $f(x,y)$ on R, the minimum occurs at the point $(1/2, -1/2)$, and there is no maximum; $f(x,y)$ tends to infinity in different directions in R.

2H-4 We have $f_x = y - 1, \quad f_y = x - 1,$ so the only critical point is at $(1, 1)$.

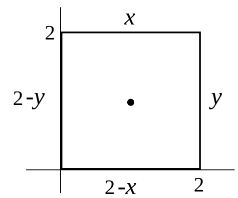
a) On the two sides of the boundary, the function $f(x, y)$ becomes respectively

$$y = 0: \quad f(x, y) \;=\; -x + 2; \qquad x = 0: \quad f(x, y) \;=\; -y + 2.$$

Since the function is linear and decreasing on both sides, it has no minimum points (informally, the minimum is $-\infty$). Since $f(1, 1) = 1$ and $f(x, x) = x^2 - 2x + 2 \to \infty$ as $x \to \infty$, the maximum of f on the first quadrant is ∞.

b) Continuing the reasoning of (a) to find the maximum and minimum points of $f(x, y)$ on the boundary, on the other two sides of the boundary square, the function $f(x, y)$ becomes

$$y = 2: \quad f(x, y) = x \qquad\qquad x = 2: \quad f(x, y) = y$$



Since $f(x, y)$ is thus increasing or decreasing on each of the four sides, the maximum and minimum points on the boundary square R can only occur at the four corner points; evaluating $f(x, y)$ at these four points, we find

$$f(0, 0) = 2; \qquad f(2, 2) = 2; \qquad\qquad f(2, 0) = 0; \qquad f(0, 2) = 0.$$

As in (a), since $f(1, 1) = 1$,

maximum points of f on R: $(0, 0)$ and $(2, 2)$; minimum points: $(2, 0)$ and $(0, 2)$.

c) The data indicates that $(1, 1)$ is probably a saddle point. Confirming this, we have $f_{xx} = 0,\ f_{xy} = 1,\ f_{yy} = 0$ for all x and y; therefore $AC - B^2 = -1 < 0$, so $(1, 1)$ is a saddle point, by the 2nd-derivative criterion.

2H-5 Since $f(x, y)$ is linear, it will not have critical points: namely, for all x and y we have $f_x = 1,\ f_y = \sqrt{3}$. So any maxima or minima must occur on the boundary circle.

We parametrize the circle by $x = \cos\theta,\ y = \sin\theta$; restricted to this boundary circle, $f(x, y)$ becomes a function of θ alone which we call $g(\theta)$:

$$g(\theta) = f(\cos\theta, \sin\theta) = \cos\theta + \sqrt{3}\sin\theta + 2.$$

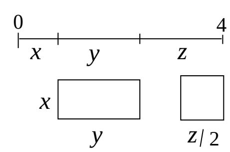
Proceeding in the usual way to find the maxima and minima of $g(\theta)$, we get

$$g'(\theta) = -\sin\theta + \sqrt{3}\cos\theta = 0, \qquad \text{or} \qquad \tan\theta = \sqrt{3}.$$

It follows that the two critical points of $g(\theta)$ are $\theta = \dfrac{\pi}{3}$ and $\dfrac{4\pi}{3}$; evaluating g at these two points, we get $g(\pi/3) = 4$ (the maximum), and $g(4\pi/3) = 0$ (the minimum).

Thus the maximum of $f(x, y)$ in the circular disc R is at $(1/2, \sqrt{3}/2)$, while the minimum is at $(-1/2, -\sqrt{3}/2)$.

2H-6 a) Since $z = 4 - x - y$, the problem is to find on R the maximum and minimum of the total area



$$f(x, y) = xy + \tfrac{1}{4}(4 - x - y)^2$$

where R is the triangle given by$R: \ 0 \le x, \quad 0 \le y, \quad x + y \le 4.$

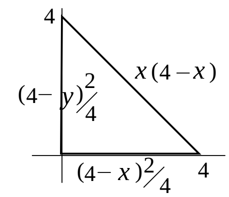
To find the critical points of $f(x, y)$, the equations $f_x = 0$ and $f_y = 0$ are respectively

$$y - \tfrac{1}{2}(4 - x - y) = 0; \quad x - \tfrac{1}{2}(4 - x - y) = 0,$$

which imply first that $x = y$, and from this, $x - \frac{1}{2}(4 - 2x)$; the unique solution is $x = 1,\ y = 1$.

The region R is a triangle, on whose sides $f(x,y)$ takes respectively the values

bottom: $y=0$; $f=\frac{1}{4}(4-x)^2$; left side: $x=0$; $f=\frac{1}{4}(4-y)^2$;
diagonal $y=4-x$; $f=x(4-x)$.

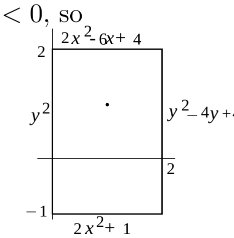


On the bottom and side, f is decreasing; on the diagonal, f has a maximum at $x=2$, $y=2$. Therefore we need to examine the three corner points and $(2,2)$ as candidates for maximum and minimum points, as well as the critical point $(1,1)$. We find

$$f(0,0)=4; \quad f(4,0)=0; \quad f(0,4)=0; \quad f(2,2)=4 \quad f(1,1)=2.$$

It follows that the critical point is just a saddle point; to get the maximum total area 4, make $x=y=0$, $z=4$, or $x=y=2$, $z=0$, either of which gives a point "rectangle" and a square of side 2; for the minimum total area 0, take for example $x=0, y=4, z=0$, which gives a "rectangle" of length 4 with zero area, and a point square.

b) We have $f_{xx}=\frac{1}{2}$, $f_{xy}=\frac{3}{2}$, $f_{yy}=\frac{1}{2}$ for all x and y; therefore $AC-B^2=-2<0$, so $(1,1)$ is a saddle point, by the 2nd-derivative criterion.



2H-7 a) $f_x=4x-2y-2$, $f_y=-2x+2y$; setting these $=0$ and solving simultaneously, we get $x=1$, $y=1$, which is therefore the only critical point.

On the four sides of the boundary rectangle R, the function $f(x,y)$ becomes:

on $y=-1$: $f(x,y) = 2x^2+1$; on $y=2$: $f(x,y)=2x^2-6x+4$
on $x=0$: $f(x,y) = y^2$; on $x=2$: $f(x,y)=y^2-4y+4$

By one-variable calculus, $f(x,y)$ is increasing on the bottom and decresing on the right side; on the left side it has a minimum at $(0,0)$, and on the top a minimum at $(\frac{3}{2},2)$. Thus the maximum and minimum points on the boundary rectangle R can only occur at the four corner points, or at $(0,0)$ or $(\frac{3}{2},2)$. At these we find:

$$f(0,-1)=1; \quad f(0,2)=4; \quad f(2,-1)=9; \quad f(2,2)=0; \quad f(\tfrac{3}{2},2)=-\tfrac{1}{2}, \quad f(0,0)=0.$$

At the critical point $f(1,1)=-1$; comparing with the above, it is a minimum; therefore,
maximum point of $f(x,y)$ on R: $(2,-1)$ minimum point of $f(x,y)$ on R: $(1,1)$

b) We have $f_{xx}=4$, $f_{xy}=-2$, $f_{yy}=2$ for all x and y; therefore $AC-B^2=4>0$ and $A=4>0$, so $(1,1)$ is a minimum point, by the 2nd-derivative criterion.

2I. Lagrange Multipliers

2I-1 Letting $P:(x,y,z)$ be the point, in both problems we want to maximize $V=xyz$, subject to a constraint $f(x,y,z)=c$. The Lagrange equations for this, in vector form, are

$$\nabla(xyz)=\lambda\cdot\nabla f(x,y,z), \qquad f(x,y,z)=c.$$

a) Here $f=c$ is $x+2y+3z=18$; equating components, the Lagrange equations become

$$yz=\lambda, \quad xz=2\lambda, \quad xy=3\lambda; \qquad x+2y+3z=18.$$

To solve these symmetrically, multiply the left sides respectively by x, y, and z to make them equal; this gives

$$\lambda x=2\lambda y=3\lambda z, \quad \text{or} \quad x=2y=3z=6, \text{ since the sum is 18.}$$

We get therefore as the answer $x = 6, \quad y = 3, \quad z = 2$. This is a maximum point, since if P lies on the triangular boundary of the region in the first octant over which it varies, the volume of the box is zero.

b) Here $f = c$ is $x^2 + 2y^2 + 4z^2 = 12$; equating components, the Lagrange equations become

$$yz = \lambda \cdot 2x, \quad xz = \lambda \cdot 4y, \quad xy = \lambda \cdot 8z; \qquad x^2 + 2y^2 + 4z^2 = 12.$$

To solve these symmetrically, multiply the left sides respectively by x, y, and z to make them equal; this gives

$$\lambda \cdot 2x^2 = \lambda \cdot 4y^2 = \lambda \cdot 8z^2, \quad \text{or} \quad x^2 = 2y^2 = 4z^2 = 4, \quad \text{since the sum is 12.}$$

We get therefore as the answer $x = 2, \quad y = \sqrt{2}, \quad z = 1$. This is a maximum point, since if P lies on the boundary of the region in the first octant over which it varies (1/8 of the ellipsoid), the volume of the box is zero.

2I-2 Since we want to minimize $x^2 + y^2 + z^2$, subject to the constraint $x^3y^2z = 6\sqrt{3}$, the Lagrange multiplier equations are

$$2x = \lambda \cdot 3x^2y^2z, \quad 2y = \lambda \cdot 2x^3yz, \quad 2z = \lambda \cdot x^3y^2; \quad x^3y^2z = 6\sqrt{3}.$$

To solve them symmetrically, multiply the first three equations respectively by x, y, and z, then divide them through respectively by 3, 2, and 1; this makes the right sides equal, so that, after canceling 2 from every numerator, we get

$$\frac{x^2}{3} = \frac{y^2}{2} = z^2; \quad \text{therefore } x = z\sqrt{3},\ y = z\sqrt{2}.$$

Substituting into $x^3y^2z = 6\sqrt{3}$, we get $3\sqrt{3}z^3 \cdot 2z^2 \cdot z = 6\sqrt{3}$, which gives as the answer, $x = \sqrt{3},\ y = \sqrt{2},\ z = 1$.

This is clearly a minimum, since if P is near one of the coordinate planes, one of the variables is close to zero and therefore one of the others must be large, since $x^3y^2z = 6\sqrt{3}$; thus P will be far from the origin.

2I-3 Referring to the solution of 2F-2, we let x be the length of the ends, y the length of the sides, and z the height, and get

$$\text{total area of cardboard } A = 3xy + 4xz + 2yz, \qquad \text{volume } V = xyz = 1.$$

The Lagrange multiplier equations $\nabla A = \lambda \cdot \nabla(xyz)$; $xyz = 1$, then become

$$3y + 4z = \lambda yz, \quad 3x + 2z = \lambda xz, \quad 4x + 2y = \lambda xy, \quad xyz = 1.$$

To solve these equations for x, y, z, λ, treat them symmetrically. Divide the first equation through by yz, and treat the next two equations analogously, to get

$$3/z + 4/y = \lambda, \quad 3/z + 2/x = \lambda, \quad 4/y + 2/x = \lambda,$$

which by subtracting the equations in pairs leads to $3/z = 4/y = 2/x$; setting these all equal to k, we get $x = 2/k, y = 4/k, z = 3/k$, which shows the proportions using least cardboard are $x : y : z = 2 : 4 : 3$.

To find the actual values of x, y, and z, we set $1/k = m$; then substituting into $xyz = 1$ gives $(2m)(4m)(3m) = 1$, from which $m^3 = 1/24$, $m = 1/2 \cdot 3^{1/3}$, giving finally

$$x = \frac{1}{3^{1/3}}, \quad y = \frac{2}{3^{1/3}}, \quad z = \frac{3}{2 \cdot 3^{1/3}}.$$

2I-4 The equations for the cost C and the volume V are $xy+4yz+6xz = C$ and $xyz = V$. The Lagrange multiplier equations for the two problems are

a) $\quad yz = \lambda(y+6z), \quad xz = \lambda(x+4z), \quad xy = \lambda(4y+6x); \qquad xy+4yz+6xz = 72$

b) $\quad y+6z = \mu\cdot yz, \quad x+4z = \mu\cdot xz, \quad 4y+6x = \mu\cdot xy; \qquad xyz = 24$

The first three equations are the same in both cases, since we can set $\mu = 1/\lambda$. Solving the first three equations in (a) symmetrically, we multiply the equations through by x, y, and z respectively, which makes the left sides equal; since the right sides are therefore equal, we get after canceling the λ,

$$xy+6xz = xy+4yz = 4yz+6xz, \quad \text{which implies} \quad xy = 4yz = 6xz.$$

a) Since the sum of the three equal products is 72, by hypothesis, we get

$$xy = 24, \quad yz = 6, \quad xz = 4;$$

from the first two we get $x = 4z$, and from the first and third we get $y = 6z$, which lead to the solution $x = 4$, $y = 6$, $z = 1$.

b) Dividing $xy = 4yz = 6xz$ by xyz leads after cross-multiplication to $x = 4z$, $y = 6z$; since by hypothesis, $xyz = 24$, again this leads to the solution $x = 4$, $y = 6$, $z = 1$.

2J. Non-independent Variables

2J-1 a) $\left(\dfrac{\partial w}{\partial y}\right)_z$ means that x is the dependent variable; get rid of it by writing $w = (z-y)^2 + y^2 + z^2 = z + z^2$. This shows that $\left(\dfrac{\partial w}{\partial y}\right)_z = 0$.

b) To calculate $\left(\dfrac{\partial w}{\partial z}\right)_y$, once again x is the dependent variable; as in part (a), we have $w = z + z^2$ and so $\left(\dfrac{\partial w}{\partial z}\right)_y = 1 + 2z$.

2J-2 a) Differentiating $z = x^2 + y^2$ w.r.t. y: $\quad 0 = 2x\left(\dfrac{\partial x}{\partial y}\right)_z + 2y$; so $\left(\dfrac{\partial x}{\partial y}\right)_z = -\dfrac{y}{x}$;

By the chain rule, $\left(\dfrac{\partial w}{\partial y}\right)_z = 2x\left(\dfrac{\partial x}{\partial y}\right)_z + 2y = 2x\left(\dfrac{-y}{x}\right) + 2y = 0.$

Differentiating $z = x^2 + y^2$ with respect to z: $\quad 1 = 2x\left(\dfrac{\partial x}{\partial z}\right)_y$; so $\left(\dfrac{\partial x}{\partial z}\right)_y = \dfrac{1}{2x}$;

By the chain rule, $\left(\dfrac{\partial w}{\partial z}\right)_y = 2x\left(\dfrac{\partial x}{\partial z}\right)_y + 2z = 1 + 2z.$

b) Using differentials, $\quad dw = 2xdx + 2ydy + 2zdz, \quad dz = 2xdx + 2ydy;$
since the independent variables are y and z, we eliminate dx by substracting the second equation from the first, which gives $\quad dw = 0\,dy + (1+2z)\,dz;$
therefore by **D2**, we get $\quad \left(\dfrac{\partial w}{\partial y}\right)_z = 0, \quad \left(\dfrac{\partial w}{\partial z}\right)_y = 1 + 2z.$

2J-3 a) To calculate $\left(\frac{\partial w}{\partial t}\right)_{x,z}$, we see that y is the dependent variable; solving for it, we get $y = \frac{zt}{x}$; using the chain rule, $\left(\frac{\partial w}{\partial t}\right)_{x,z} = x^3\left(\frac{\partial y}{\partial t}\right)_{x,z} - z^2 = x^3\frac{z}{x} - z^2 = x^2 z - z^2$.

b) Similarly, $\left(\frac{\partial w}{\partial z}\right)_{x,y}$ means that t is the dependent variable; since $t = \frac{xy}{z}$, we have by the chain rule, $\left(\frac{\partial w}{\partial z}\right)_{x,y} = -2zt - z^2\left(\frac{\partial t}{\partial z}\right)_{x,y} = -2zt - z^2 \cdot \frac{-xy}{z^2} = -zt$.

2J-4 The differentials are calculated in equation (4).

a) Since x, z, t are independent, we eliminate dy by solving the second equation for $x\,dy$, substituting this into the first equation, and grouping terms:
$dw = 2x^2y\,dx + (x^2z - z^2)dt + (x^2t - 2zt)dz$, which shows by **D2** that $\left(\frac{\partial w}{\partial t}\right)_{x,z} = x^2z - z^2$.

b) Since x, y, z are independent, we eliminate dt by solving the second equation for $z\,dt$, substituting this into the first equation, and grouping terms:
$dw = (3x^2y - zy)dx + (x^3 - zx)dy - zt\,dz$, which shows by **D2** that $\left(\frac{\partial w}{\partial z}\right)_{x,y} = -zt$.

2J-5 a) If $pv = nRT$, then $\left(\frac{\partial S}{\partial p}\right)_v = S_p + S_T \cdot \left(\frac{\partial T}{\partial p}\right)_v = S_p + S_T \cdot \frac{v}{nR}$.

b) Similarly, we have $\left(\frac{\partial S}{\partial T}\right)_v = S_T + S_p \cdot \left(\frac{\partial p}{\partial T}\right)_v = S_T + S_p \cdot \frac{nR}{v}$.

2J-6 a) $\left(\frac{\partial w}{\partial u}\right)_x = 3u^2 - v^2 - u \cdot 2v\left(\frac{\partial v}{\partial u}\right)_x = 3u^2 - v^2 - 2uv$.

$\left(\frac{\partial w}{\partial x}\right)_u = -u \cdot 2v\left(\frac{\partial v}{\partial x}\right)_u = -2uv$.

b) $dw = (3u^2 - v^2)du - 2uvdv$; $du = x\,dy + y\,dx$; $dv = du + dx$;
for both derivatives, u and x are the independent variables, so we eliminate dv, getting

$$dw = (3u^2 - v^2)du - 2uv(du + dx) = (3u^2 - v^2 - 2uv)du - 2uv\,dx,$$

whose coefficients by **D2** are $\left(\frac{\partial w}{\partial u}\right)_x$ and $\left(\frac{\partial w}{\partial x}\right)_u$.

2J-7 Since we need both derivatives for the gradient, we use differentials.

$df = 2dx + dy - 3dz$ at P; $dz = 2x\,dx + dy = 2\,dx + dy$ at P;

the independent variables are to be x and z, so we eliminate dy, getting

$df = 0\,dx - 2\,dz$ at the point $(x, z) = (1, 1)$. So $\nabla g = \langle 0, -2\rangle$ at $(1, 1)$.

2J-8 To calculate $\left(\frac{\partial w}{\partial r}\right)_\theta$, note that r and θ are independent. Therefore,
$\left(\frac{\partial w}{\partial r}\right)_\theta = \frac{\partial w}{\partial r} + \frac{\partial w}{\partial x} \cdot \left(\frac{\partial x}{\partial r}\right)_\theta$. Now, $x = r\cos\theta$, so $\left(\frac{\partial x}{\partial r}\right)_\theta = \cos\theta$. Therefore
$\left(\frac{\partial w}{\partial r}\right)_\theta = \frac{r}{\sqrt{r^2 - x^2}} + \frac{-x}{\sqrt{r^2 - x^2}} \cdot \cos\theta = \frac{r - x\cos\theta}{\sqrt{r^2 - x^2}}$

$$= \frac{r - r\cos^2\theta}{r|\sin\theta|} = \frac{r\sin^2\theta}{r|\sin\theta|} = |\sin\theta|.$$

2K. Partial Differential Equations

2K-1 $w = \frac{1}{2}\ln(x^2+y^2)$. If $(x,y) \neq (0,0)$, then

$$w_{xx} = \frac{\partial}{\partial x}(w_x) = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) = \frac{y^2-x^2}{(x^2+y^2)^2},$$

$$w_{yy} = \frac{\partial}{\partial y}(w_y) = \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right) = \frac{x^2-y^2}{(x^2+y^2)^2},$$

Therefore w satisfies the two-dimensional Laplace equation, $w_{xx} + w_{yy} = 0$; we exclude the point $(0,0)$ since $\ln 0$ is not defined.

2K-2 If $w = (x^2+y^2+z^2)^n$, then $\frac{\partial}{\partial x}(w_x) = \frac{\partial}{\partial x}\left(2x \cdot n(x^2+y^2+z^2)^{n-1}\right)$

$$= 2n(x^2+y^2+z^2)^{n-1} + 4x^2n(n-1)(x^2+y^2+z^2)^{n-2}$$

We get w_{yy} and w_{zz} by symmetry; adding and combining, we get

$$w_{xx} + w_{yy} + w_{zz} = 6n(x^2+y^2+z^2)^{n-1} + 4(x^2+y^2+z^2)n(n-1)(x^2+y^2+z^2)^{n-2}$$

$= 2n(2n+1)(x^2+y^2+z^2)^{n-1}$, which is identically zero if $n = 0$, or if $n = -1/2$.

2K-3 a) $w = ax^2 + bxy + cy^2$; $\quad w_{xx} = 2a, \quad w_{yy} = 2c.$

$$w_{xx} + w_{yy} = 0 \quad \Rightarrow \quad 2a + 2c = 0, \text{ or } c = -a.$$

Therefore all quadratic polynomials satisfying the Laplace equation are of the form

$$ax^2 + bxy - ay^2 = a(x^2-y^2) + bxy;$$

i.e., linear combinations of the two polynomials $f(x,y) = x^2 - y^2$ and $g(x,y) = xy$.

2K-4 The one-dimensional wave equation is $w_{xx} = \frac{1}{c^2}w_{tt}$. So

$$\begin{aligned} w = f(x+ct) + g(x-ct) \quad &\Rightarrow \quad w_{xx} = f''(x+ct) + g''(x-ct) \\ &\Rightarrow \quad w_t = cf'(x+ct) + -cg'(x-ct). \\ &\Rightarrow \quad w_{tt} = c^2f''(x+ct) + c^2g''(x-ct) = c^2w_{xx}, \end{aligned}$$

which shows w satisfies the wave equation.

2K-5 The one-dimensional heat equation is $w_{xx} = \frac{1}{\alpha^2}w_t$. So if $w(x,t) = \sin kxe^rt$, then

$$\begin{aligned} w_{xx} &= e^{rt} \cdot k^2(-\sin kx) = -k^2\,w. \\ w_t &= re^{rt}\sin kx = r\,w. \end{aligned}$$

Therefore, we must have $-k^2w = \frac{1}{\alpha^2}r\,w$, or $r = -\alpha^2k^2$.

However, from the additional condition that $w = 0$ at $x = 1$, we must have

$$\sin k\; e^{rt} = 0\;;$$

Therefore $\sin k = 0$, and so $k = n\pi$, where n is an integer.

To see what happens to w as $t \to \infty$, we note that since $|\sin kx| \leq 1$,

$$|w| = e^{rt}|\sin kx| \leq e^{rt}.$$

Now, if $k \neq 0$, then $r = -\alpha^2k^2$ is negative and $e^{rt} \to 0$ as $t \to \infty$; therefore $|w| \to 0$.

Thus w will be a solution satisfying the given side conditions if $k = n\pi$, where n is a non-zero integer, and $r = -\alpha^2 k^2$.