Uses of the Dot Product

1. Find the angle between the vectors $\mathbf{A} = \mathbf{i} + 8\mathbf{j}$ and $\mathbf{B} = \mathbf{i} + 2\mathbf{j}$.

Answer: As usual, call the angle in question θ. Since $\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}||\mathbf{B}| \cos\theta$ we have

$$\cos\theta = \frac{\mathbf{A} \cdot \mathbf{B}}{|\mathbf{A}||\mathbf{B}|} = \frac{\langle 1, 8\rangle \cdot \langle 1, 2\rangle}{\sqrt{65}\,\sqrt{5}} = \frac{17}{5\sqrt{13}}$$

Thus, $\theta = \cos^{-1}\left(\frac{17}{5\sqrt{13}}\right)$.

2. Take points $P = (a, 1, -1)$, $Q = (0, 1, 1)$, $R = (a, -1, 3)$. For what value(s) of a is PQR a right angle?

Answer: We need $\overrightarrow{\mathbf{QP}} \cdot \overrightarrow{\mathbf{QR}} = 0 \;\Rightarrow\; \langle a, 0, -2\rangle \cdot \langle a, -2, 2\rangle = a^2 - 4 = 0 \;\Rightarrow\; a = \pm 2$.

3. Show that the diagonals of a parallelogram are perpendicular if and only if it is a rhombus, i.e., its four sides have equal lengths.

Answer: Let two adjacent sides of the parallelogram be the vectors $\mathbf{A}$ and $\mathbf{B}$ (as shown in the figure). Then we have the two diagonals are $\mathbf{A} + \mathbf{B}$ and $\mathbf{A} - \mathbf{B}$. We have

$$(\mathbf{A} + \mathbf{B}) \cdot (\mathbf{A} - \mathbf{B}) = \mathbf{A} \cdot \mathbf{A} - \mathbf{B} \cdot \mathbf{B}.$$

Therefore,

$$(\mathbf{A} + \mathbf{B}) \cdot (\mathbf{A} - \mathbf{B}) = 0 \;\Leftrightarrow\; \mathbf{A} \cdot \mathbf{A} = \mathbf{B} \cdot \mathbf{B}.$$

I.e., the diagonals are perpendicular if and only if two adjacent edges have equal lengths. In other words, if the parallelogram is a rhombus.