Stokes' Theorem

Our text states and proves Stokes' Theorem in 12.11, but it uses the scalar form for writing both the line integral and the surface integral involved. In the applications, it is the vector form of the theorem that is most likely to be quoted, since the notations $dx \wedge dy$ and the like are not in common use (yet) in physics and engineering.

Therefore we state and prove the vector form of the theorem here. The proof is the same as in our text, but not as condensed.

Definition. Let $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ be a continuously differentiable vector field defined in an open set U of R^3. We define another vector field in U, by the equation

$$\operatorname{curl} \vec{F} = (\partial R/\partial y - \partial Q/\partial z)\,\vec{i} + (\partial P/\partial z - \partial R/\partial x)\,\vec{j} + (\partial Q/\partial x - \partial P/\partial y)\,\vec{k}.$$

We discuss later the physical meaning of this vector field.

An easy way to remember this definition is to introduce the symbolic operator "del", defined by the equation

$$\vec{\nabla} = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k},$$

and to note that $\operatorname{curl} \vec{F}$ can be evaluated by computing the symbolic determinant

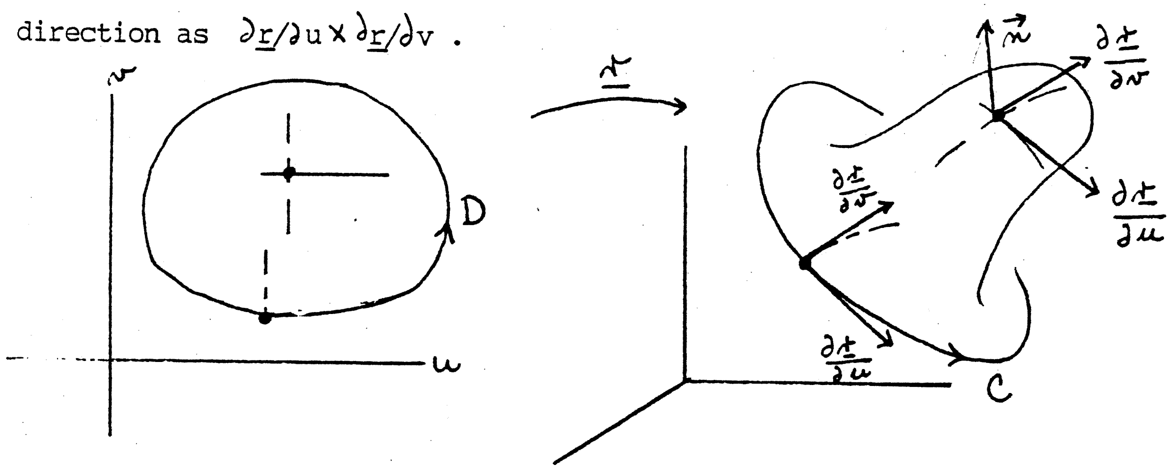
$$\operatorname{curl} \vec{F} = \vec{\nabla}\times\vec{F} = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{bmatrix}.$$

Theorem. (Stokes' theorem). Let S be a simple smooth parametrized surface in R^3, parametrized by a function $\underline{r} : T \rightarrow S$, where T is a region in the (u,v) plane. Assume that T is a Green's region, bounded by a simple closed piecewise-smooth curve D, and that $\underline{r}$ has continuous second-order partial derivatives in an open set containing T and D. Let C be the curve $\underline{r}(D)$.

If F is a continuously differentiable vector field defined in an open set of R^3 containing S and C, then

$$\int_C (\vec{F}\cdot\vec{T})\, ds = \iint_S ((\text{curl } \vec{F})\cdot\vec{n})\, dS .$$

Here the orientation of C is that derived from the counterclockwise orientation of D; and the normal $\vec{n}$ to the surface S points in the same direction as $\partial\underline{r}/\partial u \times \partial\underline{r}/\partial v$.



Remark 1. The relation between $\vec{T}$ and $\vec{n}$ is often described informally as follows: "If you walk around C in the direction specified by $\vec{T}$, with your head in the direction specified by $\vec{n}$, then the surface S is on your left." The figure indicates the correctness of this informal description.

Remark 2. We note that the equation is consistent with a change of parametrization. Suppose that we reparametrize S by taking a function $\underline{g} : W \to T$ carrying a region in the (s,t) plane onto T, and use the new parametrization $\underline{R}(s,t) = \underline{r}(\underline{g}(s,t))$. What happens to the integrals in the statement of the theorem? If $\det Dg > 0$, then the left side of the equation is unchanged, for we know that g carries the counterclockwise orientation of ∂W to the counterclockwise orientation of ∂T. Furthermore, because $\partial \underline{R}/\partial s \times \partial \underline{R}/\partial t = (\partial \underline{r}/\partial u \times \partial \underline{r}/\partial v) \det Dg$, the unit normal determined by the parametrization $\underline{R}$ is the same as that determined by $\underline{r}$, so the right side of the equation is also unchanged.

On the other hand, if $\det Dg < 0$, then the counterclockwise orientation of ∂W goes to the opposite direction on C, so that $\vec{T}$ changes sign. But in that case, the unit normal determined by $\underline{R}$ is opposite to that determined by $\underline{r}$. Thus both sides of the equation change sign.

<u>Proof of the theorem.</u> The proof consists of verifying the following three equations:

$$\int_C P\vec{i}\cdot\vec{T}\,ds = \iint_S (\partial P/\partial z\,\vec{j} - \partial P/\partial y\,\vec{k})\cdot\vec{n}\,dS,$$

$$\int_C Q\vec{j}\cdot\vec{T}\,ds = \iint_S (-\partial Q/\partial z\,\vec{i} + \partial Q/\partial x\,\vec{k})\cdot\vec{n}\,dS,$$

$$\int_C R\vec{k}\cdot\vec{T}\,ds = \iint_S (\partial R/\partial y\,\vec{i} - \partial R/\partial x\,\vec{j})\cdot\vec{n}\,dS.$$

The theorem follows by adding these equations together.

We shall in fact verify only the first equation. The others are proved similarly. Alternatively, if one makes the substitutions $\vec{i} \to \vec{j}$ and $\vec{j} \to \vec{k}$ and $\vec{k} \to \vec{i}$ and $x \to y$ and $y \to z$ and $z \to x$, then each equation is transformed into the next one. This corresponds to an orientation-preserving change of variables in R^3, so it leaves the orientations of C and S unchanged.

So let F henceforth denote the vector field $P\vec{i}$; we prove Stokes' theorem in that case.

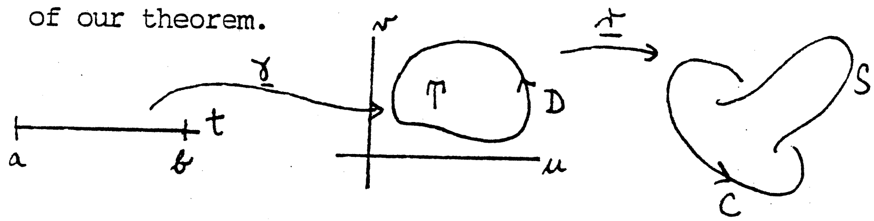
The idea of the proof is to express the line and surface integrals of the theorem as integrals over D and T, respectively, and then to apply Green's theorem to show they are equal.

Let $\underline{r}(u,v) = (X(u,v), Y(u,v), Z(u,v))$, as usual.

Choose a counterclockwise parametrization of D ; call it $\underline{\gamma}(t)$, for $a \leq t \leq b$. Then the function

$$\underline{\alpha}(t) = \underline{r}(\underline{\gamma}(t)) = (X(\underline{\gamma}(t)), Y(\underline{\gamma}(t)), Z(\underline{\gamma}(t)))$$

is the parametrization of C that we need to compute the line integral of our theorem.



We compute as follows:

$$\int_C \vec{F} \cdot d\underline{\alpha} = \int_a^b P(\underline{\alpha}(t))\, \alpha_1'(t) \text{ det}$$

$$= \int_a^b P(\underline{\alpha}(t))\left[\frac{\partial X}{\partial u}\gamma_1'(t) + \frac{\partial X}{\partial v}\gamma_2'(t)\right] dt ,$$

where $\partial X/\partial u$ and $\partial X/\partial v$ are evaluated at $\underline{\gamma}(t)$, of course. We can write this as a line integral over D. Indeed, if we let p and q be the functions

$$p(u,v) = P(\underline{r}(u,v)) \cdot \frac{\partial X}{\partial u}(u,v)$$

$$q(u,v) = P(\underline{r}(u,v)) \cdot \frac{\partial X}{\partial v}(u,v) ,$$

then our integral is just the line integral

$$\int_D (p\vec{i} + q\vec{j}) \cdot d\underline{\gamma} .$$

Now by Green's theorem, this line integral equals

(*) $$\iint_T (\partial q/\partial u - \partial p/\partial v)\, du\, dv .$$

We use the chain rule to compute the integrand. We have

$$\frac{\partial q}{\partial u} = \left(\frac{\partial P}{\partial x}\frac{\partial X}{\partial u} + \frac{\partial P}{\partial y}\frac{\partial Y}{\partial u} + \frac{\partial P}{\partial z}\frac{\partial Z}{\partial u}\right)\frac{\partial X}{\partial v} + P\frac{\partial^2 X}{\partial u \partial v}$$

$$\frac{\partial p}{\partial v} = \left(\frac{\partial P}{\partial x}\frac{\partial X}{\partial v} + \frac{\partial P}{\partial y}\frac{\partial Y}{\partial v} + \frac{\partial P}{\partial z}\frac{\partial Z}{\partial v}\right)\frac{\partial X}{\partial u} + P\frac{\partial^2 X}{\partial v \partial u} .$$

where P and its partials are evaluated at $\underline{r}(u,v)$, of course. Subtracting, we see that the first and last terms cancel each other. The double integral (*) then takes the form

$$\iint_T \left[- \frac{\partial P}{\partial y}\frac{\partial X,Y}{\partial u,v} + \frac{\partial P}{\partial z}\frac{\partial Z,X}{\partial u,v}\right] du\, dv .$$

Now we compute the surface integral of our theorem. Since $\text{curl}\, \vec{F} = \partial P/\partial z\, \vec{j} - \partial P/\partial y\, \vec{k}$, formula (12.20) on p. 435 of our text tells us we have

$$\iint_S (\text{curl}\, \vec{F})\cdot \vec{n}\, dS = \iint_T \left[\frac{\partial P}{\partial z}\frac{\partial Z,X}{\partial u,v} - \frac{\partial P}{\partial y}\frac{\partial X,Y}{\partial u,v}\right] du\, dv$$

Here $\partial P/\partial z$ and $\partial P/\partial y$ are evaluated at $\underline{r}(u,v)$, of course.

Our theorem is thus proved. □

Exercises on the divergence theorem

1. Let S be the portion of the surface $z = 9 - x^2 - y^2$ lying above the xy plane. Let $\vec{n}$ be the unit upward normal to S. Apply the divergence theorem to the solid bounded by S and the xy-plane to evaluate $\iint_S \vec{F} \cdot \vec{n}\, dS$ if:

(a) $\vec{F} = \sin(y+z)\vec{i} + e^{xz}\vec{j} + (x^2+y^2)\vec{k}$.

(b) $\vec{F} = y^2 z\vec{i} + y\vec{j} + z\vec{k}$.

Answers: (a) $81\pi/2$. (b) 81π.

2. Let S_1 denote the surface $z = 1 - x^2 - y^2$; $z \geq 0$. Let S_2 denote the unit disc $x^2 + y^2 \leq 1$, $z = 0$. Let $\vec{F} = x\vec{i} - (2x+y)\vec{j} + zk$; let $\vec{n}_1$ be the unit normal to S_1 and let $\vec{n}_2$ be the unit normal to S_2, both with positive $\vec{k}$ component. Evaluate $\iint_{S_1} \vec{F} \cdot \vec{n}_1\, dS$ and $\iint_{S_2} \vec{F} \cdot \vec{n}_2\, dS$.

Grad, Curl, Div and all that.

We study two questions about these operations:

I. Do they have natural (i.e., coordinate-free) physical or geometric interpretations?

II. What is the relation between them?

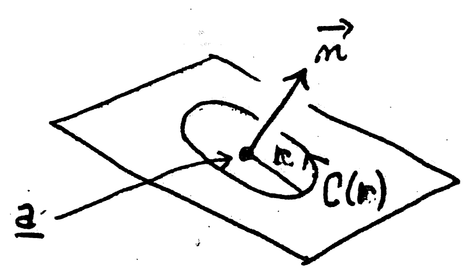
I. We already have a natural interpretation of the gradient.

For divergence, the question is answered in 12.20 of Apostol. The theorem of that section gives a coordinate-free definition of divergence $\vec{F}$, and the subsequent discussion gives a physical interpretation, in the case where $\vec{F}$ is the flux density vector of a moving fluid.

Apostol treats curl rather more briefly. Formula (12.62) on p. 461 gives a coordinate-free expression for $\vec{n} \cdot \text{curl}\, \vec{F}(\underline{a})$, as follows:

$$(*) \qquad \vec{n} \cdot \text{curl}\, \vec{F}(\underline{a}) = \lim_{r \to 0} \frac{1}{\pi r^2} \int_{C(r)} \vec{F} \cdot d\vec{\alpha}$$

where $C(r)$ is the circle of radius r centered at $\underline{a}$ lying in the plane perpendicular to $\vec{n}$ and passing through the point $\underline{a}$, and $C(r)$ is directed in a counterclockwise

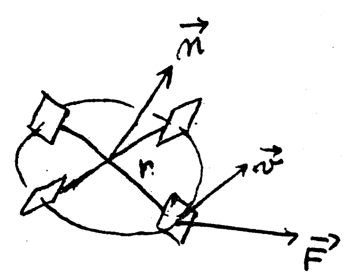


direction when viewed from the tip of $\vec{n}$. This number is called the <u>circulation of</u> $\vec{F}$ <u>at</u> <u>a</u> <u>around the vector</u> $\vec{n}$; it is clearly independent of coordinates. Then one has a coordinate-free definition of curl $\vec{F}$ as follows:

> curl $\vec{F}$ at $\underline{a}$ points in the direction of the vector around which the circulation of $\vec{F}$ is a maximum, and its magnitude equals this maximum circulation.

You will note a strong analogy here with the relation between the gradient and the directional derivative.

For a physical interpretation of curl $\vec{F}$, let us imagine $\vec{F}$ to be the velocity vector field of a moving fluid. Let us place a small paddle wheel of radius r in the fluid, with its axis along $\vec{n}$. Eventually, the paddle wheel settles



down to rotating steadily with angular speed ω (considered as positive if counterclockwise as viewed from the tip of $\vec{n}$). The tangential component $\vec{F}\cdot\vec{T}$ of velocity will tend to increase the speed ω if it is positive and to decrease ω if it is negative. On physical grounds, it is reasonable to suppose that

$$\text{average value of } (\vec{F}\cdot\vec{T}) = \text{speed of a point on one of the paddles}$$

$$= r\omega.$$

That is,

$$\frac{1}{2\pi r}\int_C \vec{F}\cdot\vec{T}\,ds = r\omega.$$

It follows that

$$\frac{1}{\pi r^2}\int_C \vec{F}\cdot\vec{T}\,ds = 2\omega,$$

so that by formula (*), we have (if r is very small),

$$\vec{n}\cdot \text{curl } \vec{F}(\underline{a}) = 2\omega.$$

In physical terms then, the vector [curl $\vec{F}(\underline{a})$] points in the direction of the axis around which our paddle wheel spins most rapidly (in a counterclockwise direction), and its magnitude equals twice this maximum angular speed.

II. What are the relations between the operations grad, curl, and div? Here is one way of explaining them.

<u>Grad</u> goes from scalar fields to vector fields, <u>Curl</u> goes from vector fields to vector fields, and <u>Div</u> goes from vector fields to scalar fields. This is summarized in the diagram:

Scalar fields		$\phi(\underline{x})$
↓	grad	
Vector fields		$\vec{F}(\underline{x})$
↓	curl	
Vector fields		$\vec{G}(\underline{x})$
↓	div	
Scalar fields		$\psi(\underline{x})$

Let us consider first the top two operations, grad and curl. We restrict ourselves to scalar and vector fields that are continuously differentiable on a region U of R^3.

Here is a theorem we have already proved:

<u>Theorem 1</u>. *$\vec{F}$ is a gradient in U if and only if $\oint_C \vec{F}\cdot d\underline{\alpha} = 0$ for every closed piecewise-smooth path in U.*

<u>Theorem 2</u>. *If $\vec{F} = \text{grad}\ \phi$ for some ϕ, then curl $\vec{F} = \vec{0}$.*

<u>Proof</u>. We compute curl $\vec{F}$ by the formula

$$\text{curl}\ \vec{F} = \det\begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ D_1 & D_2 & D_3 \\ F_1 & F_2 & F_3 \end{bmatrix}$$

$$= \vec{i}(D_2F_3 - D_3F_2) - \vec{j}(D_1F_3 - D_3F_1) + \vec{k}(D_1F_2 - D_2F_1).$$

We know that if $\vec{F}$ is a gradient, and the partials of F are continuous, then $D_i F_j = D_j F_i$ for all i, j. Hence $\text{curl}\, \vec{F} = \vec{0}$. □

__Theorem 3.__ __*If*__ $\text{curl}\, \vec{F} = \vec{0}$ __*in a star-convex region*__ U, __*then*__ $\vec{F} = \text{grad}\, \phi$ __*for some*__ ϕ __*defined in*__ U.

__*The function*__ $\psi(x) = \phi(x) + c$ __*is the most general function such that*__ $\vec{F} = \text{grad}\, \phi$.

__Proof.__ If $\text{curl}\, \vec{F} = \vec{0}$, then $D_i F_j = D_j F_i$ for all i, j. If U is star-convex, this fact implies that F is a gradient in U, by the Poincaré lemma. □

__Theorem 4.__ __*The condition*__

$$\text{curl}\, \vec{F} = \vec{0} \quad \underline{\text{in}} \quad U$$

__*does not in general imply that*__ $\vec{F}$ __*is a gradient in*__ U.

__Proof.__ Consider the vector field

$$\vec{F}(x,y,z) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0\right).$$

It is defined in the region U consisting of all of R^3 except for the z-axis. It is easy to check that $\text{curl}\, \vec{F} = \vec{0}$. To show $\vec{F}$ is not a gradient in U, we let C be the unit circle

$$\underline{\alpha}(t) = (\cos t, \sin t, 0); \quad 0 \leqslant t \leqslant 2\pi$$

in the xy-plane, and compute

$$\oint_C \vec{F} \cdot d\underline{\alpha} = 2\pi \neq 0.$$

It follows from Theorem 1 that $\vec{F}$ cannot be a gradient in U. □

Remark. A region U in R^3 is called "simply connected" if, roughly speaking, every closed curve in U bounds an orientable surface lying in U. The region R^3-(origin) is simply connected, for example, but the region R^3-(z-axis) is not.

It turns out that if U is simply connected and if curl $\vec{F} = \vec{0}$ in U, then $\vec{F}$ is a gradient in U. The proof goes roughly as follows:

Given a closed curve C in U, let S be an orientable surface in U which C bounds. Apply Stokes' theorem to that surface. One obtains the equation

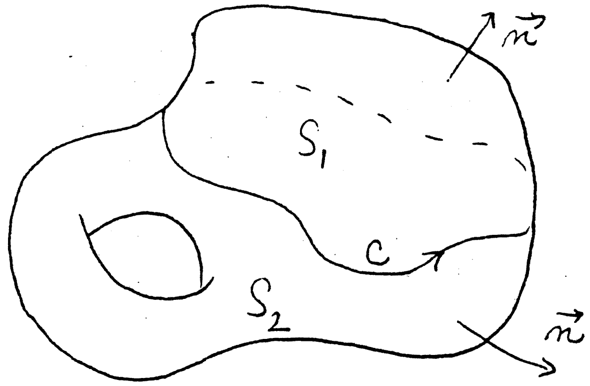
$$\oint_C \vec{F} \cdot d\underline{\alpha} = \iint \operatorname{curl} \vec{F} \cdot \vec{n}\, dS = \iint_S 0\, dS = 0.$$

Then Theorem 1 shows that $\vec{F}$ is a gradient in U.

Now let us consider the next two operations, curl and div. Again, we consider only fields that are continuously differentiable in a region U of R^3. There are analogues of all the earlier theorems:

Theorem 5. If $\vec{G}$ is a curl in U, then $\iint_S \vec{G} \cdot \vec{n}\, dS = 0$ for every orientable closed surface S in U.

Proof. Let S be a closed surface that lies in U.



(While we assume that S lies in U, we do not assume that U includes the 3-dimensional region that S bounds.) Break S up into two surfaces S_1 and S_2 that intersect in their common boundary, which is a simple smooth closed curve C. Now by hypothesis, $\vec{G} = \text{curl } \vec{F}$ for some $\vec{F}$ defined in U. We compute:

$$\iint_{S_1} \vec{G} \cdot \vec{n}\, dS = \iint_{S_1} \text{curl } \vec{F} \cdot \vec{n}\, dS = \int_C \vec{F}\ d\underline{\alpha},$$

$$\iint_{S_2} G \cdot n\, dS = \iint_{S_2} \text{curl } \vec{F} \cdot \vec{n}\, dS = -\int_C \vec{F} \cdot d\underline{\alpha}.$$

Adding, we see that

$$\iint_S \vec{G} \cdot n\, dS = 0. \square$$

Remark. The converse of Theorem 5 holds also, but we shall not attempt to prove it.

Theorem 6. If $\vec{G} = \text{curl } \vec{F}$ for some $\vec{F}$, then $\text{div } \vec{G} = 0$.

Proof. By assumption,

$$\vec{G} = \text{curl } \vec{F} = \vec{i}(D_2F_3 - D_3F_2) - \vec{j}(D_1F_3 - D_3F_1) + \vec{k}(D_1F_2 - D_2F_1).$$

Then

$$\text{div } \vec{G} = (D_1D_2F_3 - D_1D_3F_2) - (D_2D_1F_3 - D_2D_3F_1) + (D_3D_1F_2 - D_3D_2F_1)$$

$$= 0. \square$$

<u>Theorem 7.</u> <u>If</u> div $\vec{G} = 0$ <u>in a star-convex region</u> U, <u>then</u> $\vec{G} = \text{curl } \vec{F}$ <u>for some</u> $\vec{F}$ <u>defined in</u> U.

<u>The function</u> $\vec{H} = \vec{F} + \text{grad } \phi$ <u>is the most general function such that</u> $\vec{G} = \text{curl } \vec{H}$.

We shall not prove this theorem in full generality. The proof is by direct computation, as in the Poincaré lemma.

A proof that holds when U is a 3-dimensional box, or when U is all of R^3, is given in section 12.16 of Apostol. This proof also shows how to construct a specific such function $\vec{F}$ in the given cases.

Note that if $\vec{G} = \text{curl } \vec{F}$ and $\vec{G} = \text{curl } \vec{H}$, then $\text{curl}(\vec{H}-\vec{F}) = \vec{0}$. Hence by Theorem 3, $\vec{H} - \vec{F} = \text{grad } \phi$ in U, for some ϕ.

<u>Theorem 8.</u> <u>The condition</u>

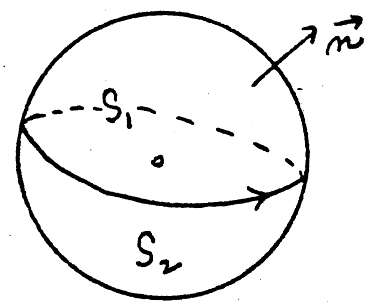
$$\text{div } \vec{G} = 0 \quad \underline{\text{in}} \quad U$$

<u>does not in general imply that</u> $\vec{G}$ <u>is a curl in</u> U.

Proof. Let $\vec{G}$ be the vector field

$$\vec{G}(x,y,z) = \frac{x\vec{i} + y\vec{j} + z\vec{k}}{(x^2+y^2+z^2)^{3/2}},$$

which is defined in the region U consisting of all of R^3 except for the origin. One readily shows by direct computation that div G = 0.



If S is the unit sphere centered at the origin, then we show that

$$\iint_S \vec{G}\cdot\vec{n}\, dA \neq 0.$$

This will imply (by Theorem 5) that $\vec{G}$ is not a curl.

If (x,y,z) is a point of S, then $\|(x,y,z)\| = 1$, so $\vec{G}(x,y,z) = x\vec{i} + y\vec{j} + z\vec{k} = \vec{n}$. Therefore

$$\iint_S \vec{G}\cdot\vec{n}\, dA = \iint_S 1\, dA = (\text{area of sphere}) \neq 0. \quad \square$$

Remark. Suppose we say that a region U in R^3 is "two-simply connected" if every closed surface in U bounds a solid region lying in U.* The region $U = R^3-(\text{origin})$ is not "two-simply connected", for example, but the region $U = R^3-(z \text{ axis})$ is.

It turns out that if U is "two-simply connected" and if $\operatorname{div} \vec{G} = 0$ in U, then $\vec{G}$ is a curl in U. The proof goes roughly as follows:

Given a closed surface S in U, let V be the region it bounds. Since $\vec{G}$ is by hypothesis defined on all of V, we can apply Gauss' theorem to compute

$$\iint_S \vec{G} \cdot \vec{n} \, dS = \iiint_V \operatorname{div} \vec{G} = \iiint_V 0 = 0.$$

Then the converse of Theorem 5 implies that $\vec{G}$ is a curl in U.

There is much more one can say about these matters, but one needs to introduce a bit of algebraic topology in order to do so. It is a bit late in the semester for that!

*The proper mathematical term for this is "homologically trivial in dimension two."