THIRD MIDTERM
MATH 18.022, MIT, AUTUMN 10

You have 50 minutes. This test is closed book, closed notes, no calculators.

Name: MODEL ANSWERS

Signature: ____________

Recitation Time: ____________

There are 5 problems, and the total number of points is 100. Show all your work. *Please make your work as clear and easy to follow as possible.*

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total	100	

1. (20pts) For what values of λ does the function $f\colon \mathbb{R}^3 \longrightarrow \mathbb{R}$,

$$f(x, y, z) = \lambda x^2 - \lambda xy + y^2 + \lambda z^2,$$

have a non-degenerate local minimum at $(0, 0, 0)$?

$$Df = (2\lambda x - \lambda y,\ 2y - \lambda x,\ 2\lambda z)$$

$$Hf = \begin{pmatrix} 2\lambda & -\lambda & 0 \\ -\lambda & 2 & 0 \\ 0 & 0 & 2\lambda \end{pmatrix}$$

$$d_1 = 2\lambda, \quad d_2 = 4\lambda \cancel{+} -\lambda^2, \quad d_3 = 2\lambda d_2$$

Minimum : $d_1 > 0$, $d_2 > 0$, $d_3 > 0$

So $\lambda > 0$, $4\lambda - \lambda^2 > 0$, $\lambda > 0$.

$$\lambda(4 - \lambda) > 0$$

$$\Downarrow$$

$$\lambda < 4.$$

$0 < \lambda < 4$. $\lambda \in (0, 4)$.

2. (20pts) Let $f\colon \mathbb{R}^3 \longrightarrow \mathbb{R}$ be the function $f(x,y,z) = x^2 - y^2 + z^2$.
(i) Show that f has a global maximum on the ellipsoid $2x^2+3y^2+z^2 = 6$.

$K = \{(x,y,z) \in \mathbb{R}^3 \mid 2x^2+3y^2+z^2=6\}$ is closed + bounded

So K is compact.

f is cts, K is compact $\Longrightarrow$ f has a global maximum.

(ii) Find this maximum.

Consider $h\colon \mathbb{R}^4 \longrightarrow \mathbb{R}$ given by

$h(x,y,z,\lambda) = x^2-y^2+z^2+\lambda(2x^2+3y^2+z^2-6)$.

Critical pts of h:

$$2x = -4\lambda x$$
$$2y = 6\lambda y$$
$$2z = -2\lambda z$$
$$2x^2+3y^2+z^2=6.$$

Either $x=y=0$, $\lambda=1$; $y=z=0$, $\lambda=-\frac{1}{2}$; $x=z=0$, $\lambda=\frac{1}{3}$

$x=y=0$, $z=\sqrt{6}$; $y=z=0$, $x=\sqrt{3}$; $x=z$, $y=\sqrt{2}$

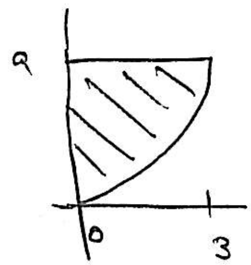
Of these three pts, ~~second~~ first gives biggest pt.

Absolute maximum is 6

3. (20pts)

(i) Switch the order of integration in the integral

$$\int_0^3 \int_{x^2}^9 x e^{-y^2}\, dy\, dx.$$



$$\int_0^9 \int_0^{\sqrt{y}} x e^{-y^2}\, dx\, dy$$

(ii) Evaluate this integral.

$$\int_0^9 e^{-y^2} \left(\int_0^{\sqrt{y}} x\, dx \right) dy = \int_0^9 e^{-y^2} \left[\frac{x^2}{2} \right]_0^{\sqrt{y}} dy$$

$$= \frac{-1}{4} \int_0^9 2y e^{-y^2}\, dy$$

$$= \frac{-1}{4} \left[e^{-y^2} \right]_0^9$$

$$= \frac{1}{4} \left(1 - e^{-81} \right).$$

4. (20pts) Let W be the region inside the sphere $x^2+y^2+z^2=1$ and inside the cone $z^2=x^2+y^2$.
Set up an integral to calculate the integral of the function yz over W and calculate this integral.

View W as a region of type 1.

$$\iiint_W zy\,dz\,dy\,dx = \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}}\left(\int_{-\sqrt{1/2-x^2}}^{\sqrt{1/2-x^2}} y\left(\int_{\sqrt{x^2+y^2}}^{\sqrt{1-x^2-y^2}} z\,dz\right)dy\right)dx$$

$$= \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}}\left(\int_{-\sqrt{1/2-x^2}}^{\sqrt{1/2-x^2}} y\left(\frac{1-x^2-y^2-x^2-y^2}{2}\right)dy\right)dx$$

$$= \int_{-1/\sqrt{2}}^{1/\sqrt{2}} 0\,dx = 0$$

as y, y^3 are odd functions.

In retrospect $J(y) = \int_{\sqrt{x^2+y^2}}^{\sqrt{1-x^2-y^2}} z\,dz$ is an even function y, so that $yJ(y)$ is an odd function. Therefore integral is zero.

5. (20pts) Let D be the region in the first quadrant bounded by the curves $x^2 - y^2 = 1$, $x^2 - y^2 = 4$, $xy = 1$ and $xy = 3$.
(i) Find $\mathrm{d}u\,\mathrm{d}v$ in terms of $\mathrm{d}x\,\mathrm{d}y$, where $u = x^2 - y^2$ and $v = xy$.

$$\frac{\partial(u,v)}{\partial(x,y)}(u,v) = \begin{vmatrix} 2x & -2y \\ y & x \end{vmatrix} = 2(x^2+y^2) \quad \frac{\partial(x,y)}{\partial(u,v)}(x,y) = \frac{1}{2(x^2+y^2)}$$

$$du\,dv = 2(x^2+y^2)\,dx\,dy$$

(ii) Evaluate the integral

$$\iint_D (x^4 - y^4)\,\mathrm{d}x\,\mathrm{d}y.$$

$$\int_1^3\int_1^4 \frac{u}{2}\,du\,dv = \frac{1}{4}\int_1^3 [u^2]_1^4\,dv$$

$$= \frac{1}{4}\int_1^3 15\,dv$$

$$= \frac{15}{2}$$