FINAL EXAM
MATH 18.022, MIT, AUTUMN 10

You have three hours. This test is closed book, closed notes, no calculators.

Name: MODEL ANSWERS

Signature:

Recitation Time:

There are 10 problems, and the total number of points is 200. Show all your work. *Please make your work as clear and easy to follow as possible.*

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
6	20	
7	20	
8	20	
9	20	
10	20	
Total	200	

1. (20pts) Find the shortest distance between the plane Π given by the equation $2x - y + 3z = 3$ and the point of intersection of the two lines l_1 and l_2 given parametrically by

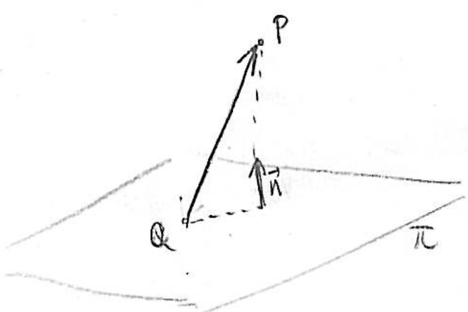
$$(x, y, z) = (2t - 3, t, 1 - t) \qquad \text{and} \qquad (x, y, z) = (1, 1 - t, t).$$

- Point where the two lines intersect: $\begin{cases} 2t_1 - 3 = 1 \\ t_1 = 1 - t_2 \\ 1 - t_1 = t_2 \end{cases} \Leftrightarrow \begin{cases} t_1 = 2 \\ t_2 = -1 \end{cases}$

so $(x, y, z) = (1, 2, -1) = P$

- Arbitrary point on the plane π; for example $Q = (0,0,1)$

Normal vector to the plane π: $\vec{n} = (2, -1, 3)$



$\vec{QP} = (1,2,-1) - (0,0,1) = (1,2,-2)$

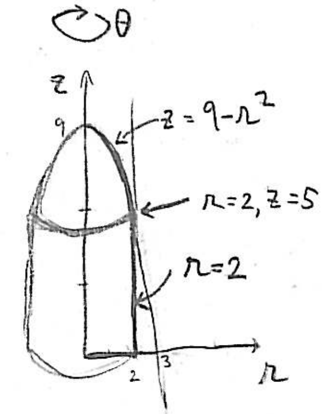
$$\text{distance} = \left\| \text{proj}_{\vec{n}} \vec{QP} \right\| = \left\| \frac{\vec{QP} \cdot \vec{n}}{\vec{n} \cdot \vec{n}} \vec{n} \right\|$$

$$= \left\| \frac{2-2-6}{4+1+9} (2,-1,3) \right\| = \frac{6}{14} \sqrt{4+1+9} = \boxed{\frac{6}{\sqrt{14}}}$$

2. (20pts) Let W be the solid bounded by the paraboloid $z = 9 - x^2 - y^2$, the xy-plane, and the cylinder $x^2 + y^2 = 4$.

(a) Set up an integral in cylindrical coordinates for evaluating the volume of W.

$$\text{vol } W = \int_0^{2\pi} \int_0^2 \int_0^{9-r^2} r \, dz \, dr \, d\theta$$



(b) Evaluate this integral.

$$\text{vol } W = 2\pi \int_0^2 r(9-r^2)\, dr$$

$$= 2\pi \int_0^2 9r - r^3)\, dr$$

$$= 2\pi \left(\frac{9r^2}{2} - \frac{r^4}{4} \right\Big|_{r=0}^{2}$$

$$= 2\pi \left(9\frac{4}{2} - \frac{16}{4} \right)$$

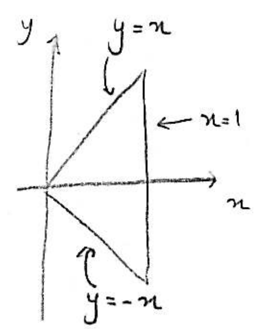
$$= 2\pi (18 - 4)$$

$$= \boxed{28\pi}$$

3. (20pts) (a) Change the order of integration of the integral

$$\int_0^1 \int_{-x}^{x} y^2 \cos(xy)\, dy\, dx. =$$

$$= \int_{-1}^{0} \int_{-y}^{1} y^2 \cos(xy)\, dx\, dy + \int_0^1 \int_y^1 y^2 \cos(xy)\, dx\, dy$$



(b) Evaluate this integral.

$$= \int_{-1}^{0} \left(y \sin(xy)\right)\Big|_{x=-y}^{1} dy + \int_0^1 \left(y \sin(xy)\right)\Big|_{x=y}^{1} dy$$

$$= \int_{-1}^{0} \left(y \sin y - y \sin(-y^2)\right) dy + \int_0^1 \left(y \sin y - y \sin(y^2)\right) dy$$

$$= \underbrace{\int_{-1}^{1} y \sin y \, dy}_{} + \underbrace{\int_{-1}^{0} -y \sin(-y^2)\, dy}_{} + \underbrace{\int_0^1 -y \sin(y^2)\, dy}_{} = ⊛$$

$$= 2\int_0^1 y \sin y \, dy$$

integration by parts

$u = y$

$v' = \sin y$

$(v = -\cos y)$

$$= 2\left[\left(-y \cos y\Big|_{y=0}^{1} - \int_0^1 -\cos y\, dy\right)\right]$$

$$= 2\left[-\cos 1 - \left(-\sin y\Big|_{y=0}^{1}\right)\right]$$

$$= 2\left[-\cos 1 + \sin 1\right]$$

$$= \left(\frac{\cos(-y)^2}{-2}\right)\Big|_{y=-1}^{0} = \frac{1-\cos 1}{-2} = \frac{\cos 1 - 1}{2}$$

3

$$= \left(\frac{\cos(-y^2)}{2}\right)\Big|_{y=0}^{1} = \frac{\cos 1 - 1}{2}$$

$$⊛ = -2\cos 1 + 2 \sin 1 + \cos 1 - 1 = \boxed{2(\sin 1) - (\cos 1) - 1}$$

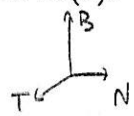
4. (20pts) Let $\vec{r}\colon I \longrightarrow \mathbb{R}^3$ be a regular smooth curve parametrised by arclength. Let $a \in I$ and suppose that

$$\vec{T}(a) = \frac{2}{7}\hat{\imath} - \frac{6}{7}\hat{\jmath} - \frac{3}{7}\hat{k}, \quad \vec{B}(a) = \frac{3}{7}\hat{\imath} - \frac{2}{7}\hat{\jmath} + \frac{6}{7}\hat{k}, \quad \frac{d\vec{N}}{ds}(a) = -\frac{13}{7}\hat{\imath} + \frac{18}{7}\hat{\jmath} - \frac{12}{7}\hat{k}.$$

Find:

(i) the unit normal vector $\vec{N}(a)$.

$B = T \times N$, so



$N = B \times T$

$$N(a) = B(a) \times T(a) = \begin{vmatrix} i & j & k \\ 3/7 & -2/7 & 6/7 \\ 2/7 & -6/7 & -3/7 \end{vmatrix} = \frac{1}{49}(6+36,\ 12+9,\ -18+4) = \boxed{\left(\frac{6}{7}, \frac{3}{7}, -\frac{2}{7}\right)}$$

(ii) the curvature $\kappa(a)$.

$N' = -\kappa T + \tau B \rightsquigarrow$ (dot with T on both sides) $\rightsquigarrow N' \cdot T = -\kappa$

$$\kappa(a) = -N'(a) \cdot T(a) = -\left(-\tfrac{13}{7}, \tfrac{18}{7}, -\tfrac{12}{7}\right) \cdot \left(\tfrac{2}{7}, \tfrac{-6}{7}, \tfrac{-3}{7}\right) = \frac{1}{49}(26+108-36) = \frac{98}{49} = \boxed{2}$$

(iii) the torsion $\tau(a)$.

$N' = -\kappa T + \tau B \rightsquigarrow$ (dot with B on both sides) $\rightsquigarrow N' \cdot B = \tau$

$$\tau(a) = N'(a) \cdot B(a) = \frac{1}{49}(-13, 18, -12) \cdot (3, -2, 6)$$

$$= \frac{1}{49}(-39 - 36 - 72)$$

$$= -\frac{147}{49} = \boxed{-3}$$

5. (20pts) Let

$$C = \{\, (x, y, z) \in \mathbb{R}^3 \mid x^2z^3 - x^3z^2 = 0, x^2y + xy^3 = 2 \,\}.$$

(a) Show that in a neighbourhood of the point $P = (1, 1, 1)$, C is a smooth curve with a parametrisation of the form

$$\vec{g}(x) = (x, g_1(x), g_2(x)).$$

Let $F(x,y,z) = (x^2z^3 - x^3z^2,\ x^2y + xy^3 - 2)$

$$\frac{\partial F}{\partial(y,z)}(1,1,1) = \det\begin{bmatrix} 0 & 3x^2z^2 - 2x^3z \\ x^2 + 3xy^2 & 0 \end{bmatrix}\Bigg|_{(x,y,z)=(1,1,1)} = \det\begin{bmatrix} 0 & 1 \\ 4 & 0 \end{bmatrix} = -4 \neq 0$$

Then, by the implicit function theorem, in a neighborhood of $P=(1,1,1)$, C can be parametrized by x.

(b) Find a parametrisation of the tangent line to C at P.

Take the derivative with respect to x on both sides of the equation $F(x, g_1(x), g_2(x)) = (0,0)$, and evaluate at $x=1$:

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} g_1'(x) + \frac{\partial F}{\partial z} g_2'(x) = (0,0) \qquad \leftarrow \text{actually two equations!}$$

$$\Leftrightarrow \begin{cases} (2xz^3 - 3x^2z^2) + 0 + (3x^2z^2 - 2x^3z)g_2'(x) = 0 \\ (2xy + y^3) + (x^2 + 3xy^2)\, g_1'(x) = 0 \end{cases} \Leftrightarrow \begin{cases} -1 + g_2'(1) = 0 \\ 3 + 4g_1'(1) = 0 \end{cases} \Leftrightarrow \begin{cases} g_2'(1) = 1 \\ g_1'(1) = -\frac{3}{4} \end{cases}$$

Vector tangent to curve C at $P=(1,1,1)$: $(1, g_1'(1), g_2'(1)) = (1, -\tfrac{3}{4}, 1)$

Point on tangent line: $P = (1,1,1)$

Equation of the line: $(x,y,z) = (1,1,1) + t(1, -\tfrac{3}{4}, 1)$

or $\boxed{(x,y,z) = (1+t,\ 1-\tfrac{3}{4}t,\ 1+t)}$

6. (20pts) Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be the function $f(x, y) = xy$.
(a) Show that f has a global maximum on the ellipse $9x^2 + 4y^2 = 36$.

$$\left.\begin{array}{l} f \text{ is a continuous function} \\ \text{the ellipse is a compact set} \\ \text{(bounded + closed)} \end{array}\right\} \Longrightarrow f \text{ has a global maximum on the ellipse}$$

(b) Find this global maximum value of f.

$$\nabla f = \lambda \nabla g \Leftrightarrow (y, x) = \lambda(18x, 8y) \Leftrightarrow \begin{cases} y = 18\lambda x \\ x = 8\lambda y \end{cases}$$

$$y = 18 \cdot 8 \lambda^2 y \quad \begin{cases} y = 0 \longrightarrow \cancel{x = y = 0} \quad (\text{not on ellipse}) \\ \lambda = \pm \frac{1}{12} \end{cases}$$

$\lambda = \frac{1}{12}$: $y = \frac{18}{12} x = \frac{3}{2} x$, so $9x^2 + 4 \frac{9}{4} x^2 = 36 \Leftrightarrow 18x^2 = 36$
$\Leftrightarrow x = \pm\sqrt{2} \rightsquigarrow y = \frac{3}{2} x = \pm \frac{3}{\sqrt{2}}$

$$f\left(\pm\sqrt{2}, \pm \frac{3}{\sqrt{2}}\right) = 3$$

$\lambda = -\frac{1}{12}$: $y = -\frac{18}{12} x = -\frac{3}{2} x$, so $x = \pm\sqrt{2} \rightsquigarrow y = \mp \frac{3}{\sqrt{2}}$

$$f\left(\pm\sqrt{2}, \mp \frac{3}{\sqrt{2}}\right) = -3$$

Maximum value of f is $\boxed{3}$ (at points $\left(\sqrt{2}, \frac{3}{\sqrt{2}}\right)$ and $\left(-\sqrt{2}, -\frac{3}{\sqrt{2}}\right)$)

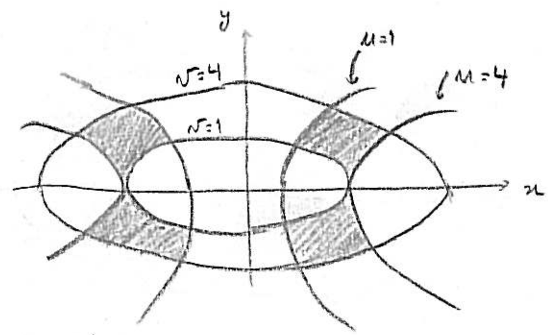
7. (20pts) Let D be the region bounded by the four curves $x^2 - y^2 = 1$, $x^2 - y^2 = 4$, $x^2/4 + y^2 = 1$ and $x^2/16 + y^2/4 = 1$.
(a) Compute $dx\,dy$ in terms of $du\,dv$, where $u = x^2 - y^2$ and $v = x^2/4 + y^2$.

$$\frac{\partial(u,v)}{\partial(x,y)} = \left| \det \begin{bmatrix} 2x & -2y \\ \frac{2x}{4} & 2y \end{bmatrix} \right| = |4xy + xy| = 5|xy|$$

So $\boxed{dx\,dy = \frac{1}{5|xy|}\,du\,dv}$

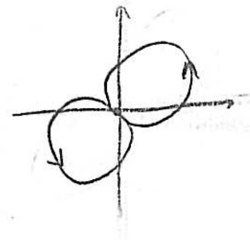
(b) Evaluate the integral

$$\iint_D \frac{xy}{y^2 - x^2}\,dx\,dy.$$



The function $\frac{xy}{y^2-x^2}$ is odd with respect to x (and y too), so its integral over the first and fourth quadrants will cancel out with the integral over the second and third quadrants:

$$\iint_D \frac{xy}{y^2-x^2}\,dx\,dy = 0$$



8. (20pts) (a) Find the area of the region that lies inside the closed curve defined by the equation $r = 2a(1 + \sin 2\theta)$ in polar coordinates

$$\text{area} = \int_0^{2\pi} \int_0^{2a(1+\sin 2\theta)} r\,dr\,d\theta = \int_0^{2\pi} \frac{4a^2(1+\sin 2\theta)^2}{2}\,d\theta$$

$$= 2a^2 \int_0^{2\pi} (1+\sin 2\theta)^2\,d\theta = 2a^2 \int_0^{2\pi} \left(1 + (\sin 2\theta)^2 + 2\sin 2\theta\right) d\theta$$

$$= 2a^2 \left(2\pi + \pi + \left(-\cos 2\theta\Big|_{\theta=0}^{2\pi}\right)\right) = \boxed{6\pi a^2}$$

$$\boxed{\int_0^{2\pi} (\sin 2\theta)^2\,d\theta = \frac{1}{2}\int_0^{2\pi} (\sin 2\theta)^2 + (\cos 2\theta)^2\,d\theta = \frac{1}{2}\,2\pi = \pi}$$

(b) Find the line integral of $\vec{F} = -y\hat{\imath} + x\hat{\jmath}$ along the curve, oriented counter-clockwise.

Green's theorem in vector form (Stokes theorem

$$\oint_C \vec{F}\cdot d\vec{s} = \iint (\nabla\times\vec{F})\cdot\hat{k}\,dA$$

$$= \iint 2\,dA = 2\cdot 6\pi a^2 = \boxed{12\pi a^2}$$

$$\nabla\times\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & k \\ \partial_x & \partial_y & \partial_z \\ -y & x & 0 \end{vmatrix} = (0,0,2)$$

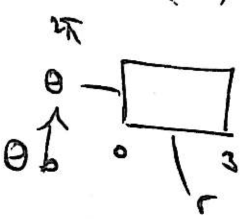
9. (20pts) Let S be the circle with centre $(2, 3, -1)$ and radius 3 lying in the plane with normal vector $\hat{n} = (\frac{1}{3}, \frac{2}{3}, -\frac{2}{3})$. Find the flux of the vector field $\vec{F}(x, y, z) = y\hat{\jmath} + z\hat{\jmath} + x\hat{k}$ through S in the direction of $\hat{n}$.

First find two vectors $\hat{a}, \hat{b}$ orthogonal to $\hat{n}$.

$\hat{a} = \left(-\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right) \quad \hat{a}\cdot\hat{n} = 0$ ✓

$$\hat{b} = \hat{n}\times\hat{a} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1/3 & 2/3 & -2/3 \\ -2/3 & 2/3 & 1/3 \end{vmatrix} = \frac{2}{3}\hat{\imath} + \frac{\hat{\jmath}}{3} + \frac{2\hat{k}}{3}$$

$(\hat{a}, \hat{b}, \hat{n})$ right handed set. $\quad U = [0,3]\times(0, 2\pi)$



$\hat{g} : U \longrightarrow \mathbb{R}^3$

$(r,\theta) \longrightarrow \vec{OP} + r\cos\theta\,\hat{a} + r\sin\theta\,\hat{b}$

$\frac{d\hat{g}}{dr} = \cos\theta\,\hat{a} + \sin\theta\,\hat{b} \qquad \frac{d\hat{g}}{d\theta} = -r\sin\theta\,\hat{a} + r\cos\theta\,\hat{b}$

$$\frac{d\hat{g}}{dr}\times\frac{d\hat{g}}{d\theta} = (\cos\theta\,\hat{a} + \sin\theta\,\hat{b})\times(-r\sin\theta\,\hat{a} + r\cos\theta\,\hat{b})$$

$$= r(\cos^2\theta + \sin^2\theta)\,\hat{n} = r\hat{n}.$$

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_U \vec{F}\cdot\left(\frac{\partial\hat{g}}{\partial r}\times\frac{\partial\hat{g}}{\partial\theta}\right) ds = \int_0^{2\pi}\int_0^3 \left(\vec{F}(r,\theta)\cdot r\hat{n}\right) dr\, d\theta$$

$$-9\pi. = \int_0^{2\pi}\left[\frac{-r^2}{2}\right]_0^3 d\theta = \int_0^{2\pi}\int_0^3 (\alpha r + \beta r\cos\theta + \gamma\sin\theta)\, dr\, d\theta$$

$\int_0^{2\pi}\cos\theta\, d\theta = \int_0^{2\pi}\sin\theta\, d\theta = 0$

$\alpha = \frac{3}{3} - \frac{2}{3} - \frac{2}{3} = -1.$

ignore β, γ.

10. (20pts) Let $S_a(P)$ denote the sphere centred at P of radius a and oriented outwards. A smooth vector field $\vec{F}$ is defined on all of $\mathbb{R}^3$ except the three points $P_1 = (0,0,0)$, $P_2 = (4,0,0)$ and $P_3 = (8,0,0)$. Suppose that the divergence of $\vec{F}$ is zero and that

$$\iint_{S_1(P_1)} \vec{F}\cdot \mathrm{d}\vec{S} = 1, \qquad \iint_{S_6(P_1)} \vec{F}\cdot \mathrm{d}\vec{S} = 3 \qquad \text{and} \qquad \iint_{S_6(P_3)} \vec{F}\cdot \mathrm{d}\vec{S} = 5.$$

Find the following flux integrals:

(a)

$$\iint_{S_1(P_2)} \vec{F}\cdot \mathrm{d}\vec{S}. = 2$$

Let M be the closed ball of radius 6 containing P_1, minus the interior of the balls of radius 1 about P_1 and P_2. Then $\partial M = S_6(P_1) + S_1'(P_1) + S_1'(P_2)$ and M is a smooth 2-manifold with boundary

Gauss divergence thm:

$$0 = \iiint_M \nabla F \, dV = \iint_{\partial M} F.d\vec{s}$$

$$= \iint_{S_6(P_1)} \vec{F}.d\vec{s} + \iint_{S_1'(P_1)} \vec{F}.d\vec{s} + \iint_{S_1'(P_2)} \vec{F} d\vec{s}$$

$$\iint_{S_1(P_2)} \vec{F}.d\vec{s} = \iint_{S_6(P_1)} \vec{F}.d\vec{s} - \iint_{S_1(P_1)} \vec{F} d\vec{s}$$

$$= 3 - 1$$

$$= 2$$

(b)

$$\iint_{S_1(P_3)} \vec{F}\cdot \mathrm{d}\vec{S}.$$

same reasoning:

$$= 5 - 2 = 3.$$