FINAL EXAM
MATH 18.022, MIT, AUTUMN 10

You have three hours. This test is closed book, closed notes, no calculators.

Name:____________________

Signature:____________________

Recitation Time:____________________

There are 10 problems, and the total number of points is 200. Show all your work. *Please make your work as clear and easy to follow as possible.*

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
6	20	
7	20	
8	20	
9	20	
10	20	
Total	200	

1. (20pts) Find the shortest distance between the plane Π given by the equation $2x - y + 3z = 3$ and the point of intersection of the two lines l_1 and l_2 given parametrically by

$$(x, y, z) = (2t - 3, t, 1 - t) \qquad \text{and} \qquad (x, y, z) = (1, 1 - t, t).$$

2. (20pts) Let W be the solid bounded by the paraboloid $z = 9-x^2-y^2$, the xy-plane, and the cylinder $x^2 + y^2 = 4$.
(a) Set up an integral in cylindrical coordinates for evaluating the volume of W.

(b) Evaluate this integral.

3. (20pts) (a) Change the order of integration of the integral

$$\int_0^1 \int_{-x}^{x} y^2 \cos(xy)\, dy\, dx.$$

(b) Evaluate this integral.

4. (20pts) Let $\vec{r}\colon I \longrightarrow \mathbb{R}^3$ be a regular smooth curve parametrised by arclength. Let $a \in I$ and suppose that

$$\vec{T}(a) = \frac{2}{7}\hat{\imath} - \frac{6}{7}\hat{\jmath} - \frac{3}{7}\hat{k}, \quad \vec{B}(a) = \frac{3}{7}\hat{\imath} - \frac{2}{7}\hat{\jmath} + \frac{6}{7}\hat{k}, \quad \frac{d\vec{N}}{ds}(a) = -\frac{13}{7}\hat{\imath} + \frac{18}{7}\hat{\jmath} - \frac{12}{7}\hat{k}.$$

Find:

(i) the unit normal vector $\vec{N}(a)$.

(ii) the curvature $\kappa(a)$.

(iii) the torsion $\tau(a)$.

5. (20pts) Let

$$C = \{ (x, y, z) \in \mathbb{R}^3 \,|\, x^2 z^3 - x^3 z^2 = 0, x^2 y + x y^3 = 2 \}.$$

(a) Show that in a neighbourhood of the point $P = (1, 1, 1)$, C is a smooth curve with a parametrisation of the form

$$\vec{g}(x) = (x, g_1(x), g_2(x)).$$

(b) Find a parametrisation of the tangent line to C at P.

6. (20pts) Let $f\colon \mathbb{R}^2 \longrightarrow \mathbb{R}$ be the function $f(x,y) = xy$.
(a) Show that f has a global maximum on the ellipse $9x^2 + 4y^2 = 36$.

(b) Find this global maximum value of f.

7. (20pts) Let D be the region bounded by the four curves $x^2 - y^2 = 1$, $x^2 - y^2 = 4$, $x^2/4 + y^2 = 1$ and $x^2/16 + y^2/4 = 1$.

(a) Compute $dx\,dy$ in terms of $du\,dv$, where $u = x^2 - y^2$ and $v = x^2/4 + y^2$.

(b) Evaluate the integral

$$\iint_D \frac{xy}{y^2 - x^2}\,dx\,dy.$$

8. (20pts) (a) Find the area of the region that lies inside the closed curve defined by the equation $r = 2a(1 + \sin 2\theta)$ in polar coordinates

(b) Find the line integral of $\vec{F} = -y\hat{\imath} + x\hat{\jmath}$ along the curve, oriented counter-clockwise.

9. (20pts) Let S be the circle with centre $(2, 3, -1)$ and radius 3 lying in the plane with normal vector $\hat{n} = (\frac{1}{3}, \frac{2}{3}, -\frac{2}{3})$. Find the flux of the vector field $\vec{F}(x, y, z) = y\hat{\jmath} + z\hat{\jmath} + x\hat{k}$ through S in the direction of $\hat{n}$.

10. (20pts) Let $S_a(P)$ denote the sphere centred at P of radius a and oriented outwards. A smooth vector field $\vec{F}$ is defined on all of $\mathbb{R}^3$ except the three points $P_1 = (0,0,0)$, $P_2 = (4,0,0)$ and $P_3 = (8,0,0)$. Suppose that the divergence of $\vec{F}$ is zero and that

$$\iint_{S_1(P_1)} \vec{F}\cdot d\vec{S} = 1, \qquad \iint_{S_6(P_1)} \vec{F}\cdot d\vec{S} = 3 \qquad \text{and} \qquad \iint_{S_6(P_3)} \vec{F}\cdot d\vec{S} = 5.$$

Find the following flux integrals:

(a)

$$\iint_{S_1(P_2)} \vec{F}\cdot d\vec{S}. = 2$$