18.02 Multivariable Calculus
Fall 2007

CV. Changing Variables in Multiple Integrals

1. Changing variables.

Double integrals in x, y coordinates which are taken over circular regions, or have integrands involving the combination $x^2 + y^2$, are often better done in polar coordinates:

$$\iint_R f(x,y)\,dA = \iint_R g(r,\theta)\,r\,dr d\theta \;. \tag{1}$$

This involves introducing the new variables r and θ, together with the equations relating them to x, y in both the forward and backward directions:

$$r = \sqrt{x^2+y^2}, \qquad \theta = \tan^{-1}(y/x); \qquad\qquad x = r\cos\theta, \qquad y = r\sin\theta \;. \tag{2}$$

Changing the integral to polar coordinates then requires three steps:

A. Changing the integrand $f(x,y)$ to $g(r,\theta)$, by using (2);

B. Supplying the area element in the r, θ system: $dA = \; r\,dr\,d\theta$;

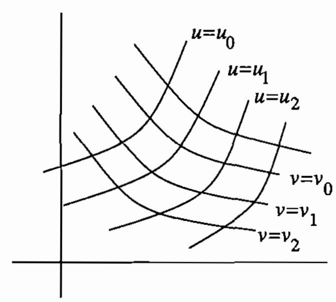
C. Using the region R to determine the limits of integration in the r, θ system.

In the same way, double integrals involving other types of regions or integrands can sometimes be simplified by changing the coordinate system from x, y to one better adapted to the region or integrand. Let's call the new coordinates u and v; then there will be equations introducing the new coordinates, going in both directions:

$$u = u(x,y), \qquad v = v(x,y); \qquad\qquad x = x(u,v), \qquad y = y(u,v) \tag{3}$$

(often one will only get or use the equations in one of these directions). To change the integral to u, v-coordinates, we then have to carry out the three steps $\mathbf{A}, \mathbf{B}, \mathbf{C}$ above. A first step is to picture the new coordinate system; for this we use the same idea as for polar coordinates, namely, we consider the grid formed by the level curves of the new coordinate functions:

$$u(x,y) = u_0, \qquad v(x,y) = v_0 \;. \tag{4}$$



Once we have this, algebraic and geometric intuition will usually handle steps **A** and **C**, but for **B** we will need a formula: it uses a determinant called the **Jacobian**, whose notation and definition are

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} \;. \tag{5}$$

Using it, the formula for the area element in the u, v-system is

$$dA = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv \;, \tag{6}$$

so the change of variable formula is

$$\iint_R f(x,y)\,dx\,dy \;=\; \iint_R g(u,v)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv\ , \tag{7}$$

where $g(u,v)$ is obtained from $f(x,y)$ by substitution, using the equations (3).

We will derive the formula (5) for the new area element in the next section; for now let's check that it works for polar coordinates.

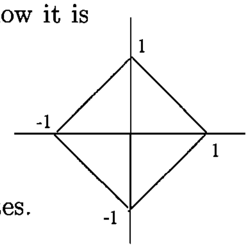
Example 1. Verify (1) using the general formulas (5) and (6).

Solution. Using (2), we calculate:

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r(\cos^2\theta + \sin^2\theta) = r\ ,$$

so that $dA = r\,dr\,d\theta$, according to (5) and (6); note that we can omit the absolute value, since by convention, in integration problems we always assume $r \geq 0$, as is implied already by the equations (2).

We now work an example illustrating why the general formula is needed and how it is used; it illustrates step **C** also — putting in the new limits of integration.



Example 2. Evaluate $\displaystyle\iint_R \left(\frac{x-y}{x+y+2}\right)^2 dx\,dy$ over the region R pictured.

Solution. This would be a painful integral to work out in rectangular coordinates. But the region is bounded by the lines

$$x+y=\pm 1, \qquad x-y=\pm 1 \tag{8}$$

and the integrand also contains the combinations $x-y$ and $x+y$. These powerfully suggest that the integral will be simplified by the change of variable (we give it also in the inverse direction, by solving the first pair of equations for x and y):

$$u = x+y, \qquad v = x-y; \qquad\qquad x = \frac{u+v}{2}, \qquad y = \frac{u-v}{2}\ . \tag{9}$$

We will also need the new area element; using (5) and (9) above. we get

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix} = -\frac{1}{2}\ ; \tag{10}$$

note that it is the second pair of equations in (9) that were used, not the ones introducing u and v. Thus the new area element is (this time we do need the absolute value sign in (6))

$$dA = \frac{1}{2}\,du\,dv\ . \tag{11}$$

We now combine steps **A** and **B** to get the new double integral; substituting into the integrand by using the first pair of equations in (9), we get

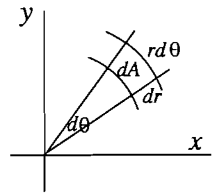
$$\iint_R \left(\frac{x-y}{x+y+2}\right)^2 dx\,dy = \iint_R \left(\frac{v}{u+2}\right)^2 \frac{1}{2}\,du\,dv\ . \tag{12}$$

In uv-coordinates, the boundaries (8) of the region are simply $u = \pm 1$, $v = \pm 1$, so the integral (12) becomes

$$\iint_R \left(\frac{v}{u+2}\right)^2 \frac{1}{2}\,du\,dv = \int_{-1}^{1}\int_{-1}^{1} \left(\frac{v}{u+2}\right)^2 \frac{1}{2}\,du\,dv$$

We have

$$\text{inner integral} = -\frac{v^2}{2(u+2)}\Big]_{u=-1}^{u=1} = \frac{v^2}{3}\,; \qquad \text{outer integral} = \frac{v^3}{9}\Big]_{-1}^{1} = \frac{2}{9}\,.$$

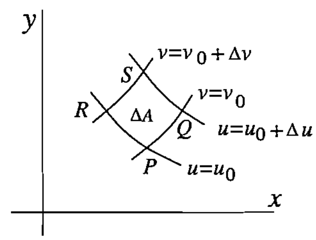


2. The area element.

In polar coordinates, we found the formula $dA = r\,dr\,d\theta$ for the area element by drawing the grid curves $r = r_0$ and $\theta = \theta_0$ for the r,θ-system, and determining (see the picture) the infinitesimal area of one of the little elements of the grid.

For general u, v-coordinates, we do the same thing. The grid curves (4) divide up the plane into small regions ΔA bounded by these contour curves. If the contour curves are close together, they will be approximately parallel, so that the grid element will be approximately a small parallelogram, and

$$\text{(13)} \qquad \Delta A \approx \text{area of parallelogram PQRS} = |PQ \times PR|$$



In the uv-system, the points P, Q, R have the coordinates

$$P : (u_0, v_0), \qquad Q : (u_0 + \Delta u, v_0), \qquad R : (u_0, v_0 + \Delta v)\ ;$$

to use the cross-product however in (13), we need PQ and PR in $\mathbf{i}\,\mathbf{j}$- coordinates. Consider PQ first; we have

$$\text{(14)} \qquad PQ = \Delta x\,\mathbf{i} + \Delta y\,\mathbf{j}\ ,$$

where Δx and Δy are the changes in x and y as you hold $v = v_0$ and change u_0 to $u_0 + \Delta u$. According to the definition of partial derivative,

$$\Delta x \approx \left(\frac{\partial x}{\partial u}\right)_0 \Delta u, \qquad \Delta y \approx \left(\frac{\partial y}{\partial u}\right)_0 \Delta u;$$

so that by (14),

$$\text{(15)} \qquad PQ \approx \left(\frac{\partial x}{\partial u}\right)_0 \Delta u\ \mathbf{i} + \left(\frac{\partial y}{\partial u}\right)_0 \Delta u\ \mathbf{j}\ .$$

In the same way, since in moving from P to R we hold u fixed and increase v_0 by Δv,

$$\text{(16)} \qquad PR \approx \left(\frac{\partial x}{\partial v}\right)_0 \Delta v\ \mathbf{i} + \left(\frac{\partial y}{\partial v}\right)_0 \Delta v\ \mathbf{j}\ .$$

We now use (13); since the vectors are in the xy-plane, $PQ \times PR$ has only a $\mathbf{k}$-component, and we calculate from (15) and (16) that

$$\mathbf{k}\text{-component of } PQ \times PR \approx \begin{vmatrix} x_u\Delta u & y_u\Delta u \\ x_v\Delta v & y_v\Delta v \end{vmatrix}_0$$

$$\text{(17)} \qquad = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}_0 \Delta u \Delta v\ ,$$

where we have first taken the transpose of the determinant (which doesn't change its value), and then factored the Δu and Δv out of the two columns. Finally, taking the absolute value, we get from (13) and (17), and the definition (5) of Jacobian,

$$\Delta A \approx \left|\frac{\partial(x,y)}{\partial(u,v)}\right|_0 \Delta u \Delta v \ ;$$

passing to the limit as $\Delta u, \Delta v \to 0$ and dropping the subscript 0 (so that P becomes any point in the plane), we get the desired formula for the area element,

$$dA = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\, dv \ .$$

3. Examples and comments; putting in limits.

If we write the change of variable formula as

$$\iint_R f(x,y)\, dx\, dy = \iint_R g(u,v) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\, dv \ , \tag{18}$$

where

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} , \qquad g(u,v) = f(x(u,v), y(u,v)), \tag{19}$$

it looks as if the essential equations we need are the inverse equations:

$$x = x(u,v), \qquad y = y(u,v) \tag{20}$$

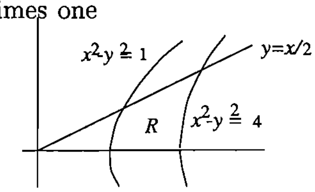
rather than the direct equations we are usually given:

$$u = u(x,y), \qquad v = v(x,y) \ . \tag{21}$$

If it is awkward to get (20) by solving (21) simultaneously for x and y in terms of u and v, sometimes one can avoid having to do this by using the following relation (whose proof is an application of the chain rule, and left for the Exercises):

$$\frac{\partial(x,y)}{\partial(u,v)} \frac{\partial(u,v)}{\partial(x,y)} = 1 \tag{22}$$

The right-hand Jacobian is easy to calculate if you know $u(x,y)$ and $v(x,y)$; then the left-hand one — the one needed in (19) — will be its reciprocal. Unfortunately, it will be in terms of x and y instead of u and v, so (20) still ought to be needed, but sometimes one gets lucky. The next example illustrates.



Example 3. Evaluate $\displaystyle\iint_R \frac{y}{x}\, dx\, dy$, where R is the region pictured, having as boundaries the curves $x^2 - y^2 = 1$, $\quad x^2 - y^2 = 4$, $\quad y = 0$, $\quad y = x/2$.

Solution. Since the boundaries of the region are contour curves of $x^2 - y^2$ and y/x, and the integrand is y/x, this suggests making the change of variable

$$u = x^2 - y^2, \qquad v = \frac{y}{x} \ . \tag{23}$$

We will try to get through without solving these backwards for x, y in terms of u, v. Since changing the integrand to the u, v variables will give no trouble, the question is whether we can get the Jacobian in terms of u and v easily. It all works out, using (22):

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 2x & -2y \\ -y/x^2 & 1/x \end{vmatrix} = 2 - 2y^2/x^2 = 2 - 2v^2; \qquad \text{so} \quad \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{2(1-v^2)},$$

according to (22). We use now (18), put in the limits, and evaluate; note that the answer is positive, as it should be, since the integrand is positive.

$$\begin{aligned} \iint_R \frac{y}{x}\, dx\, dy &= \iint_R \frac{v}{2(1-v^2)}\, du\, dv \\ &= \int_0^{1/2}\int_1^4 \frac{v}{2(1-v^2)}\, du\, dv \\ &= -\frac{3}{4}\ln(1-v^2)\Big]_0^{1/2} = -\frac{3}{4}\ln\frac{3}{4}. \end{aligned}$$

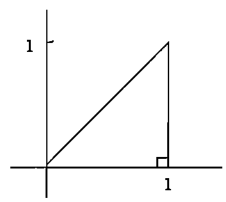
Putting in the limits

In the examples worked out so far, we had no trouble finding the limits of integration, since the region R was bounded by contour curves of u and v, which meant that the limits were constants.

If the region is not bounded by contour curves, maybe you should use a different change of variables, but if this isn't possible, you'll have to figure out the uv-equations of the boundary curves. The two examples below illustrate.

Example 4. Let $u = x + y$, $v = x - y$; change $\int_0^1\int_0^x dy\, dx$ to an iterated integral $du\, dv$.

Solution. Using (19) and (22), we calculate $\dfrac{\partial(x,y)}{\partial(u.v)} = -1/2$, so the Jacobian factor in the area element will be $1/2$.



To put in the new limits, we sketch the region of integration, as shown at the right. The diagonal boundary is the contour curve $v = 0$; the horizontal and vertical boundaries are not contour curves — what are their uv-equations? There are two ways to answer this; the first is more widely applicable, but requires a separate calculation for each boundary curve.

Method 1 *Eliminate x and y from the three simultaneous equations $u = u(x,y)$, $v = v(x,y)$, and the xy-equation of the boundary curve.* For the x-axis and $x = 1$, this gives

$$\begin{cases} u = x + y \\ v = x - y \\ y = 0 \end{cases} \Rightarrow \quad u = v; \qquad \begin{cases} u = x + y \\ v = x - y \\ x = 1 \end{cases} \Rightarrow \begin{cases} u = 1 + y \\ v = 1 - y \end{cases} \Rightarrow \quad u + v = 2.$$

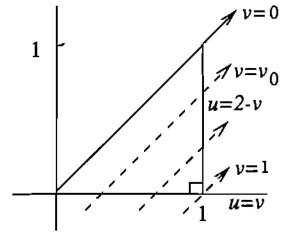
Method 2 *Solve for x and y in terms of u, v; then substitute $x = x(u,v)$, $y = y(u,v)$ into the xy-equation of the curve.*

Using this method, we get $x = \frac{1}{2}(u+v)$, $y = \frac{1}{2}(u-v)$; substituting into the xy-equations:

$$y = 0 \Rightarrow \frac{1}{2}(u - v) = 0 \Rightarrow u = v; \qquad x = 1 \Rightarrow \frac{1}{2}(u + v) = 1 \Rightarrow u + v = 2.$$

To supply the limits for the integration order $\iint du\,dv$, we

1. first hold v fixed, let u increase; this gives us the dashed lines shown;
2. integrate with respect to u from the u-value where a dashed line enters R (namely, $u = v$), to the u-value where it leaves (namely, $u = 2 - v$).
3. integrate with respect to v from the lowest v-values for which the dashed lines intersect the region R (namely, $v = 0$), to the highest such v-value (namely, $v = 1$).



Therefore the integral is $\displaystyle\int_0^1 \int_v^{2-v} \frac{1}{2}\, du\, dv$.

(As a check, evaluate it, and confirm that its value is the area of R. Then try setting up the iterated integral in the order $dv\, du$; you'll have to break it into two parts.)

Example 5. Using the change of coordinates $u = x^2 - y^2$, $v = y/x$ of Example 3, supply limits and integrand for $\displaystyle\iint_R \frac{dxdy}{x^2}$, where R is the infinite region in the first quadrant under $y = 1/x$ and to the right of $x^2 - y^2 = 1$.

Solution. We have to change the integrand, supply the Jacobian factor, and put in the right limits.

To change the integrand, we want to express x^2 in terms of u and v; this suggests eliminating y from the u, v equations; we get

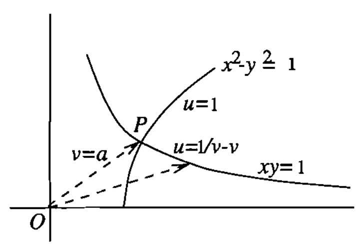
$$u = x^2 - y^2, \quad y = vx \quad \Rightarrow \quad u = x^2 - v^2x^2 \quad \Rightarrow \quad x^2 = \frac{u}{1 - v^2} .$$

From Example 3, we know that the Jacobian factor is $\dfrac{1}{2(1 - v^2)}$; since in the region R we have by inspection $0 \le v < 1$, the Jacobian factor is always positive and we don't need the absolute value sign. So by (18) our integral becomes

$$\iint_R \frac{dx\,dxy}{x^2} = \iint_R \frac{1 - v^2}{2u(1 - v^2)}\, du\, dv = \iint_R \frac{du\, dv}{2u}$$

Finally, we have to put in the limits. The x-axis and the left-hand boundary curve $x^2 - y^2 = 1$ are respectively the contour curves $v = 0$ and $u = 1$; our problem is the upper boundary curve $xy = 1$. To change this to $u - v$ coordinates, we follow Method 1:

$$\begin{cases} u = x^2 - y^2 \\ y = vx \\ xy = 1 \end{cases} \Rightarrow \begin{cases} u = x^2 - 1/x^2 \\ v = 1/x^2 \end{cases} \Rightarrow u = \frac{1}{v} - v .$$



The form of this upper limit suggests that we should integrate first with respect to u. Therefore we hold v fixed, and let u increase; this gives the dashed ray shown in the picture; we integrate from where it enters R at $u = 1$ to where it leaves, at $u = \dfrac{1}{v} - v$.

The rays we use are those intersecting R: they start from the lowest ray, corresponding to $v = 0$, and go to the ray $v = a$, where a is the slope of OP. Thus our integral is

$$\int_0^a \int_1^{1/v - v} \frac{du\, dv}{2u} .$$

To complete the work, we should determine a explicitly. This can be done by solving $xy = 1$ and $x^2 - y^2 = 1$ simultaneously to find the coordinates of P. A more elegant approach is to add $y = ax$ (representing the line OP) to the list of equations, and solve all three simultaneously for the slope a. We substitute $y = ax$ into the other two equations, and get

$$\begin{cases} ax^2 = 1 \\ x^2(1-a^2) = 1 \end{cases} \quad \Rightarrow \quad a = 1 - a^2 \quad \Rightarrow \quad a = \frac{-1+\sqrt{5}}{2}\ ,$$

by the quadratic formula.

4. Changing coordinates in triple integrals

Here the coordinate change will involve three functions

$$u = u(x,y,z), \quad v = v(x,y,z) \quad w = w(x,y,z)$$

but the general principles remain the same. The new coordinates u, v, and w give a three-dimensional grid, made up of the three families of contour surfaces of u, v, and w. Limits are put in by the kind of reasoning we used for double integrals. What we still need is the formula for the new volume element dV.

To get the volume of the little six-sided region ΔV of space bounded by three pairs of these contour surfaces, we note that nearby contour surfaces are approximately parallel, so that ΔV is approximately a parallelepiped, whose volume is (up to sign) the 3×3 determinant whose rows are the vectors forming the three edges of ΔV meeting at a corner. These vectors are calculated as in section 2; after passing to the limit we get

(24) $$dV = \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw\ ,$$

where the key factor is the **Jacobian**

(25) $$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix}\ .$$

As an example, you can verify that this gives the correct volume element for the change from rectangular to spherical coordinates:

(26) $$x = \rho\sin\phi\cos\theta, \qquad y = \rho\sin\phi\sin\theta, \qquad z = \rho\cos\phi;$$

while this is a good exercise, it will make you realize why most people prefer to derive the volume element in spherical coordinates by geometric reasoning.

Exercises: Section 3D