Review of Taylor's Series

Professor Jerison was away for this lecture, so Professor Haynes Miller took his place.

A *power series* or *Taylor's series* is a way of writing a function as a sum of integral powers of x:

$$f(x) = a_0 + a_x x + a_2 x^2 + \cdots$$

Polynomials are power series; they go on for a finite number of terms and then end, so that all of the a_j equal 0 after a certain point. Since polynomials are a special type of power series, it's not surprising that power series behave almost exactly like polynomials.

Given a power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ there is a number R $(0 \leq R \leq \infty)$ for which, when $|x| < R$, the sum $\sum_{n=0}^{\infty} a_n x^n$ converges and when $|x| > R$ the sum diverges. R is called the *radius of convergence*.

For $|x| < R$, $f(x)$ has all its higher derivatives, and Taylor's formula tells us that $a_n = \frac{f^{(n)}(0)}{n!}$. So:

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots$$

Whenever you write out a power series you should say what the radius of convergence is. The radius of convergence of this series is infinity; in other words, the series converges for any value of x.

Example: (Due to Leonhard Euler) e^x

We know that if $f(x) = e^x$ then $f^{(n)}(x) = e^x$ for all n, and so $f^{(n)}(0) = 1$. Applying Taylor's formula we see that:

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots, \quad R = \infty.$$

Question: How many terms of the series do we need to write out?

Answer: Write out enough terms so that you can see what the pattern is.

Question: What functions can be written as power series?

Answer: Any function that has a reasonable expression can be written as a power series. This is not a very precise answer because the true answer is a little bit complicated. For now, it's enough that any of the functions that occur in calculus (like sines cosines, and tangents) all have power series expansions.