

**PROFESSOR:** Welcome back to recitation. In this video, I want us to compute the following limit. It's the limit as  $n$  goes to infinity of the sum for  $i$  equals 0 to  $n$  minus 1 of the following,  $\frac{2}{n}$  times the quantity  $2^i$  over  $n$  quantity squared minus 1. Now this might look a little intimidating to try and take a limit of this, but what I'd like you to do, as a hint to you, is that you should think about this as being potentially a Riemann sum of a certain function.

So if you can figure out the function, and you can figure out the appropriate interval that you're taking a Riemann sum over, as  $n$  goes to infinity, you should be able to write this as an integral. We know how to use the fundamental theorem of calculus to determine that a definite integral in many cases. Hopefully this is a function for which we know a way to do that.

So that's my hint to you. Think about it, it's a Riemann sum approximating an integral, and I'll give you a while to work on it, and then I'll be back.

OK, welcome back. Well hopefully it's been fun for you to look at this problem so far. Let me just remind you what we were doing. We were trying to compute a limit as  $n$  goes to infinity of the sum from  $i$  equals 0 to  $n$  minus 1 of  $\frac{2}{n}$  times  $2^i$  over  $n$  squared minus 1.

So I gave you the big hint that this is probably going to be written as an integral. So let me show you some pieces of this sum that should help us see what the integral is, and then I'll make a guess about what this is, and then I'll try to give an educated way to check my guess.

So the first thing we noticed is that there is one thing, this is a product of two functions, and one of them-- well, of  $n$ , I guess. But this is a product of two things. One thing appears in every single term that you have for  $i$ . So the sum has  $n$  terms, and they're all going to be  $\frac{2}{n}$  times this, and the  $i$  is going to change. But this does not change, this  $\frac{2}{n}$  does not change, right? In fact, I could even pull that out if I wanted. But, I don't want to pull it out of the sum right now. I want us to look at what's actually going on in this product.

So if this thing is appearing over and over again, and we know this is probably a Riemann sum, then chances are this is our  $\Delta x$ . So  $\Delta x$  being equal to  $\frac{2}{n}$ , we know  $\Delta x$  equals  $b$  minus  $a$  over  $n$ , where  $b$  and  $a$  are our left endpoint-- oh, sorry, our right endpoint-- and our left endpoint. Right? We integrate from  $a$  to  $b$ . So  $b$  minus  $a$  is the length of the interval. So this is really dividing up whatever interval we're integrating over, into  $n$  equal subintervals. So that's my first thought, is that  $b$  minus  $a$  over  $n$  is equal to  $\frac{2}{n}$ .

And now we want to try and figure out, well, what the heck is this. Well, when we take a Riemann sum, remember when we take a Riemann sum what we get. We get the sum of  $\Delta x$  times  $f$  of  $x$  sub  $i$ , and  $i$  is what's varying from 0 to  $n$  minus 1. Let me put a little curve in here so we see those are two different things.

So this is  $i$  equals 0 to  $n$  minus 1. I have this  $\Delta x$  here. I'm anticipating this is some  $f$  of  $x$  sub  $i$ . And so the question is, what  $f$  is it? Right? If I know what  $f$  it is, then I know that this sum will be equal to something, the integral from  $a$  to  $b$  of  $f$  of  $x$   $dx$ , and  $a$  and  $b$  will differ by 2. So that's where I'm heading. So now the question is, what is this a function of? What function is this, I should say.

Now first guess would be something like, well, I'm taking some quantity, I'm squaring it, and subtracting 1. So my first guess for this function is  $x$  squared minus 1. I mean, that seems easy to me. Let's see if this would actually even make sense just by looking at the subscripts, or sorry, the index, the indices I have here.

So what do I have? Well, when I put in  $i$  equals 0-- let's put down some of these values-- when I put in  $i$  equals 0, I get 2 times 0 over  $n$  squared minus 1. When I put in  $i$  equals 1, I get 2 times 1 over  $n$  squared minus 1. And I go all the way up, to 2 times  $n$  minus 1 over  $n$  squared minus 1. So it's kind of a long sum there, but these are, this is what our sum of these things looks like if I pull out the 2 over  $n$ .

So here I get 0 squared minus 1. That looks pretty good. Here I get 2 times 1 over  $n$  squared minus 1. So it does look like I'm doing something, taking something, squaring it, subtracting 1. Does it make sense that these are the kind of  $x$  values I would expect to get if this were the Riemann sum of  $x$  squared minus 1? It does, and let's think about why.

I'm starting at  $x$  equals 0 here, it sure looks like. Let's look at what happens when I go all the way over here. What happens when  $n$  gets really, really big, is it this ratio approaches 2. So it's 2 times  $n$  minus 1 over  $n$ .  $n$  minus 1 over  $n$ , as  $n$  gets arbitrarily large, as  $n$  gets really big, then this approaches 2. So this is approaching 2 squared minus 1. So it's giving me more evidence that this is probably the function  $x$  squared minus 1. And now I'm starting to believe the interval is 0 to 2. I know it's a length 2 interval, and it's looking like the interval is 0 to 2.

Let's come back and talk about one more thing. The one other thing that you should notice is that how does this value differ from this value, and the next, and the next, and the next? They

differ by  $\frac{2}{n}$ . So each time whatever input I had previously, I'm now adding  $\frac{2}{n}$  to the next input. And that should make sense of what we know about Riemann sums, because what I do, is I divide my interval into these subintervals of length  $\frac{2}{n}$ . I'm evaluating it at one x-value, that-- I'm starting, in this case, at 0. Then the next interval is  $\frac{2}{n}$  more. Then I evaluate at that x-value. The next one is  $\frac{2}{n}$  more, and I evaluate at that x-value.

So this is looking like-- I'm going to write it here, this is my guess-- integral from 0 to 2 of  $x^2 - 1$  dx. And now to make myself feel good about this-- I'm pretty sure it's that. To make you feel good about this, I'm going to divide this into four subintervals, and I'm going to show you what the Riemann sum with four intervals looks like, and then we can talk about how it relates to this one over here.

OK, so let me draw a graph. Actually, I'll use just white chalk again. Let me draw a graph of  $x^2 - 1$  from 0 to 2. So 0, 1, 2, minus 1. OK, so at 0,  $x^2 - 1$  is negative 1. At  $x$  equals 1,  $x^2 - 1$  is 0. And at 2,  $x^2 - 1$  is 3. So hopefully, this is all going to go into the video, and-- in the video screen, I mean. And there we go, something like that. So this is, you know, it continues over here, but I'm really only interested in this part.

So now let's look at what the subintervals are. And now I'm going to get some colored chalk. So what are the subintervals? I'm taking  $\frac{1}{4}$ , OK? And so  $\Delta x$ , in this case, is  $\frac{2}{4}$ , which is equal to  $\frac{1}{2}$ . Right?

And so what are my, what are-- so what is my sum going to look like? Well, I am going to tell you that I'm also going to use left-handed endpoints. And I mentioned earlier why that is, I believe. Maybe I didn't. But, I started off at  $i$  equals 0, and my first input value was 0. My last input value had an  $n - 1$  in it instead of an  $n$ . So I was doing, somehow, the second-to-last place that I was interested in here. So it's definitely more of a left-hand endpoint thing. So I'm going to do this with left-hand endpoints. And I'm not going to simplify as I go. I'm going to write it out in sort of an expanded form.

OK, so let's write it out in expanded form. So the Riemann sum-- this is  $y$  equals  $x^2 - 1$ . The Riemann sum is, the first term is  $\frac{1}{2}$  times what? It's the value, this x-value, which is 0, evaluated on this curve, so  $0^2 - 1$ . The next term-- I'll just write them right below each other-- is  $\frac{1}{2}$ . 'Cause again, let's draw a picture of what the first one is, sorry. It's this rectangle. Right? It's evaluated-- It's length  $\frac{1}{2}$ , and it's the function evaluated at 0.

The next one is length  $\frac{1}{2}$ , and it's going to be the function evaluated at whatever this left-

hand endpoint is here. So it's going to be this area. So it's going to be length  $1/2$ , and then the height is going to be at  $x$  equals  $1/2$ , so  $1/2$  quantity squared minus 1. The next one is going to be this interval. Well, there's no rectangle to draw because it's just, the output is zero at the left endpoint here. So it's just going to be-- it's going to have output equal to 0, at length  $1/2$  and height 0. But we'll write it out anyway. It's going to be  $1/2$  times the quantity-- now, I went up  $1/2$  more, so it's going to be two  $1/2$ 's, two times  $1/2$  squared minus 1.

Let me just show you why I did this. OK, if we look at the picture, here I'd gone up  $1/2$  from my initial value. Here I'd gone up another  $1/2$  from my initial value. So that's two  $1/2$ 's from my initial value of 0. The next one is going to be three  $1/2$ 's, so this is three  $1/2$ 's away, or commonly known as  $3/2$ . OK?

So that one is going to be--  $1/2$  is the base length again, times the quantity 3 times  $1/2$  squared minus 1. And that is in the picture, this rectangle.

Great. So what are we see here? If we look at this, these four pieces, what do we have? We have a  $1/2$  in front each time. Which, what was the  $1/2$ ? It was  $b$  minus  $a$  over  $n$ . So  $b$  minus  $a$  was 2,  $n$  was 4. So maybe I should have kept that as 2 over 4. But, it's a little easier to write it as  $1/2$  because of what I'm doing next.

I square something, I subtract 1. I go up by the value that this is from the initial one here. And so now I'm taking the output of what was in here. I now take the output at  $1/2$  more than what was here. Now I take it at two  $1/2$ 's more than what was here, or  $1/2$  more than what was there, and then three  $1/2$ 's more than what was here, or one more than what was there. That's kind of confusing, but let's go back to the picture and see what it is.

My  $\Delta x$  was  $1/2$ , right? So I evaluate at the first place, and I evaluate one more up, and then I evaluate one more up, and I evaluate one more up, which gives me outputs here, there, there, and there. Right?

So really if you go back and you look at the formulation of the sum, this was  $2$  over  $n$  times quantity  $2i$  over  $n$  squared minus 1, you can see the  $2$  over  $n$  is my  $1/2$ , and then this is maybe the hardest part to see, but that's the  $2$  over  $n$  is my  $1/2$  again, and the  $i$  is this thing that's coming in as 1, 2, 3. And so that  $i$  was going from 0 to  $n$  minus 1-- so I should have said 0, 1, 2, 3. Right? So that  $i$  is the 0 to  $n$  minus 1, and then I'm evaluating that, and then I add them all up. So when I take the sum, I get, for  $n$  equals 4, I get this. So in fact, this is just a guess, still maybe you should, maybe you should convince yourself more.

I'm actually convinced at this point, based on not just this evidence, but the evidence I understood before about how the function works. Maybe you want to compare it when  $n$  equals 6, or something like that. You may need a little more evidence to make you understand this particular piece. But, hopefully that makes sense to you that this is really just  $i$  times  $\Delta x$ , and then evaluated somewhere. That's the main idea of this component.

OK, well hopefully this is informative to you. If you want to know the exact answer of how to compute the sum, obviously you just take the integral. So I know you can do that. So that's where I'll stop.