$\int \sin^4(x)\cos^2(x)\,dx$

Compute $\int \sin^4(x)\cos^2(x)\,dx$.

Solution

Because all of the exponents in this problem are even, our chosen solution involves half angle formulas:

$$\cos^2\theta = \frac{1+\cos(2\theta)}{2}$$
$$\sin^2\theta = \frac{1-\cos(2\theta)}{2}.$$

Because we have to do a lot of writing before we actually integrate anything, we'll start with some "side work" to convert the integrand into something we know how to integrate.

$$\begin{aligned}
\sin^4 x\cos^2 x &= (\sin^2 x)^2\cos^2 x \\
&= \left(\frac{1-\cos(2x)}{2}\right)^2\left(\frac{1+\cos(2x)}{2}\right) \\
&= \left(\frac{1-2\cos(2x)+\cos^2(2x)}{4}\right)\left(\frac{1+\cos(2x)}{2}\right) \\
&= \frac{1-2\cos(2x)+\cos^2(2x)+\cos(2x)-2\cos^2(2x)+\cos^3(2x)}{8} \\
&= \frac{1-\cos(2x)-\cos^2(2x)+\cos^3(2x)}{8}
\end{aligned}$$

This is all the side work we need to do here, because we know that:

$$\begin{aligned}
\int \cos^2(2x) &= \frac{x}{2}+\frac{\sin(2x)}{4}+c_1 \qquad \text{and} \\
\int \cos^3(2x) &= \frac{1}{2}\sin(2x)-\frac{1}{6}\sin^3(2x)+c_2.
\end{aligned}$$

We conclude that:

$$\begin{aligned}
\int \sin^4 x\cos^2 x\,dx &= \int \frac{1-\cos(2x)-\cos^2(2x)+\cos^3(2x)}{8}\,dx \\
&= \frac{1}{8}\left[x-\frac{1}{2}\sin(2x)-\left(\frac{x}{2}+\frac{\sin(2x)}{4}+c_1\right)\right. \\
&\qquad \left.+\left(\frac{1}{2}\sin(2x)-\frac{1}{6}\sin^3(2x)+c_2\right)\right] \\
&= \frac{1}{8}\left[\frac{x}{2}-\frac{\sin(2x)}{4}-\frac{1}{6}\sin^3(2x)\right]+C
\end{aligned}$$

$$= \frac{x}{16} - \frac{\sin(2x)}{32} - \frac{\sin^3(2x)}{48} + C$$

It's difficult to check that this is the correct answer. If $C = 0$ this is an odd function which is at least consistent with the integrand being an even function.