PROFESSOR: Welcome back to recitation. In this video I want us to work on the following problem. A very shallow circular reflecting pool has uniform depth D and radius R. And this is in meters. And a disinfecting chemical is released at its center. After a few hours of symmetrical diffusion outward, the concentration at a point little r meters from the center is k over 1 plus r squared grams per cubic meter. The k here is a constant. So we drop some chemical into the middle of the pool. And then it diffuses outward. That's the idea there.

Now the question is the following. What amount of chemical was released into the pool? And obviously the amount will be in grams. Now to give you a hint, probably you want to draw a picture of this. And then maybe get some estimates or some approximations using shells. Or you might just say, you know, some strips, some shells probably is a better word for it. And then you should think about the fact that, as you get better and better estimates, your approximation should tend toward an integral.

So with that hint, I'll give you a little time to work on this. And when we come back, I'll show you how I do it.

OK, welcome back. Hopefully you were able to make some headway. And so let me start off by doing what I asked you to do, which is draw a picture. Now the first picture is fairly simple. The first picture is my pool. Right? But that's not quite enough to tell me some estimates. But what do we have? So the pool has radius capital R and depth D. And when I said we want to do some approximation, what I really meant is we want to-- let me actually get another color here-- we want to say, take some fixed radius out and assume that-- let me actually draw this behind-- some fixed radius out, and assume that the diffusion is constant for some small bit. Some small strip like this. Which, actually, you notice it's going to be rotated around. Because all of this is relative to distance from the center.

So, I'm hoping that you can see kind of what this drawing is doing. Essentially what we have here is if we open that up, this blue cylindrical shell is approximately a piece-- oh it doesn't quite look flat-- but, a piece that looks like-- oh well we'll just stick with that-- a little prism here. So that's approximately, if I were to cut this blue cylindrical shell here open and lay it down flat, it would be approximately a piece kind of like this. Right?

And so, what we're going to do is estimate first what amount of chemical is released based on pieces that look like this. And then we're going to let those pieces get very, very narrow and get more and more of them. And this should remind you of Riemann sums. And how Riemann sums, as you let the number of things you're summing over tend to 0-- sorry-- tend to infinity, so that the little pieces are getting narrower and narrower, you're actually going to end up with an integral. So this is where we're headed.

So let's just make sure we understand kind of all the pieces that are happening. What we're going to do is we're going to take a bunch of these cylinders, and let's just determine that we'll take n of them. I shouldn't say cylinder. Sorry, I should say shells. We're going to take n of these shell-type things. So I'm going to say the radii-- I'm going to start with r_0 equals 0. And I'm going to take n different radii. So r_0, r_1, up to-- sorry this is 0-- so r_n is equal to capital R. So I guess I'm taking n plus 1, but 0 is not really a radius. But I have n different partitions.

For each partition of this big cylinder, I get a piece like this. Right? I get a piece that, when I open it up, looks approximately like this. Now, what I want is a total amount. I want grams. Right? And what I'm given-- if we come back over here-- what I'm given is the concentration at a certain radius. It's k over 1 plus r squared grams per cubic meter. Now, if nothing else, you should have looked at this problem and seen, well if I want grams and I have something in grams per cubic meter, somewhere I'm going to need something with cubic meters to cancel this unit, so I end up with grams. So if nothing else, then maybe you can you can think, oh I need to understand volume of something in order to solve this problem. Right?

Now if we have n different partitions-- so n shells-- that all started off as this sort of blue-type thing and I open up and look like this. Then I want to figure out what is the volume of these shells. Once I know the volume of that shell, I can figure out the amount, roughly, of chemical in that piece by multiplying by the concentration.

So let's figure out what this volume is in terms of these little radii I'm looking at. Well, when I open it up, what do I get? We're assuming this here is this little segment here, and so that's our delta r, that's our change in radius. That's how much I'm changing. So this would be some r subscript i and this would be some r subscript i plus 1. And let's assume that we're taking everything from the smaller radius. We're going to do everything from the smaller radius. So then, when I open this up, this circle is going to be my length. So my length is 2 pi r sub i. And then the height is easy. The height is constant. The height is just capital D. Right?

So the volume of each shell-- let me come over here-- the volume of a shell is something like 2 pi r sub i times D times delta r. And so then, the amount of chemical in the shell is going to be the volume times the concentration. Right? The volume times the concentration. So the volume is, again, this. I'm going to put the D in front of the r sub i. 2 pi D r sub i delta r times the concentration, which if we come back over here, the concentration is k divided by 1 plus r squared grams per cubic meter. The r in this case is the r sub i. I'm assuming, because I'm approximating this, that everywhere in the shell has the same concentration, has the concentration of the interior radius.

So if we come back over here, we're going to write k over 1 plus r sub i squared. And now what do I do to estimate the amount in the entire pool? Well I add all of these up. So let me come over here and write down what the sum will look like.

So I'm going to be summing from i equals-- I said I was taking the interior radius, I think-- i equals 0 to n minus 1 of this quantity. 2*pi*D-- let me put the k in there as well-- k. And then r sub i over 1 plus r sub i squared delta r. So this is our approximation of the amount of chemical in the pool.

And again, we always want to check and make sure. I didn't write in any units, but do the units make sense? Well we know the units make sense because when I did the amounts in the shell, I did volume times concentration. And volume times concentration is going to be in grams. This is in cubic meters. This is in grams per cubic meter. So I know I have the right unit. So that's a good way to check. It doesn't guarantee you've done it correctly. But at least you can check and make sure you didn't do it, you know that-- how would I say this? I would say that if the units are not in grams, you know you did something wrong. So at least now we know, OK, it passes the first smell test.

Now what do I do to find the exact value? Well what I want to do is, I want to come back over to the picture I have here and I want to let these shells get smaller and smaller. And how do I let those shells get smaller and smaller? Narrower and narrower, I should say. I let them get narrower by increasing the number of radii on which I do this kind of operation. So I'm coming over here and now I'm letting the n get bigger and bigger. And as n gets bigger and bigger, these values are still determined the same way. But over here this n is getting larger and larger.

So I can take the limit, as n goes to infinity, of this quantity to get the exact amount. What is the limit as n goes to infinity of this? This is actually an integral. It's the integral from 0 to capital R-- because my radius is ranging from 0 to that big R-- of this exact function of R.

Right?

So I'm going to put the 2*pi*D*k out here. And then I get little r over 1 plus r squared and the delta r becomes our dr. So this is, in fact, going to be the amount, in grams, of the chemical that was released into the pool.

So we've set up our integral. I think I'll stop here. If you want to go further and determine it, you can. And you may want to think about what strategy, obviously, what strategy you want to use in order to solve this problem. I'll give you a hint. Maybe the best way to solve this problem is the fact that when you take the derivative of 1 plus r squared, you actually get 2r. And so that derivative is almost up here. So maybe this is a good hint to give you how you would continue this problem.

But I'll stop there. So let me just go back one more time and remind you what we were doing. What we were doing, if we come back over here, is we were given a situation where we knew a certain function of the radius, the distance from the center. And we wanted to determine the total amount of chemical that was released into the pool. And so we estimated. We figured out a way to estimate it in terms of splitting up the radii. We had the radius from 0 to big R. And we just divided up the radii and assumed certain things were constant in these regions.

And we determined the right function to find-- if we move over here, over here-- we would determine the right function to find the amount of chemical in a certain shell, assuming that the concentration was constant throughout that shell. And then, what we do is we know that if we let those shells get arbitrarily narrow, that means that we're letting the number of radii over which we're doing this go to infinity. And we know that this summation that we have here, as n goes to infinity, becomes an integral. So that's really, we exploited what we know about this sum and letting this partition, letting the delta r get arbitrarily small. That that, in the limit, goes to an integral. And then that's something we can definitively calculate. So I think I will stop there.