Limits

Last class we talked about a series of secant lines approaching the "limit" of a tangent line, and about how as Δx approaches zero, $\frac{\Delta y}{\Delta x}$ approaches the "limit" $y' = \frac{dy}{dx}$. Now we want to talk about limits more carefully; this will include some of our first steps towards our goal of being able to differentiate every function you know.

Some limits are easy to compute:

$$\lim_{x \to 3} \frac{x^2 + x}{x + 1} = \frac{3^2 + 3}{3 + 1} = \frac{12}{4} = 3$$

With an easy limit, you can get a meaningful answer just by plugging in the limiting value. This is because when x is close to 3, the value of the function $f(x) = \frac{x^2+x}{x+1}$ is close to $f(3)$.

Some limits are not easy to compute. For example, the definition of the derivative:

$$\lim_{x \to x_0} \frac{\Delta f}{\Delta x} = \lim_{x \to x_0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

is never an easy limit, because the denominator $\Delta x = 0$ is not allowed. (The limit $x \to x_0$ is computed under the implicit assumption that $x \neq x_0$.) We'll always need to cancel Δx before we can make sense out of the limit.

Other "hard" limits would be:

$$\lim_{x \to -1} \frac{x^2 + x}{x + 1} \quad \text{and} \quad \lim_{x \to \infty} \frac{x^2 + x}{x + 1}.$$

Any limit involving infinity or division by zero is going to be harder to compute; sometimes the answer will will be that there is no limit.

To complete our discussion of limits, we need just one more piece of notation — the concepts of left hand and right hand limits.

The limit

$$\lim_{x \to x_0^+} f(x)$$

is known as the *right-hand limit* and means that you should use values of x that are greater than x_0 (to the right of x_0 on the number line) to compute the limit. Shown below is the graph of the function:

$$f(x) = \begin{cases} x + 1 & x > 0 \\ -x & x \leq 0 \end{cases}$$

The right-hand limit $\lim_{x \to 0^+} f(x)$ equals 1.

The *left-hand limit*

$$\lim_{x \to x_0^-} f(x)$$

is found by looking at values of $f(x)$ when x is less than x_0 (to the left of x_0 on the number line). For this function, $\lim_{x \to 0^-} f(x) = 0$.

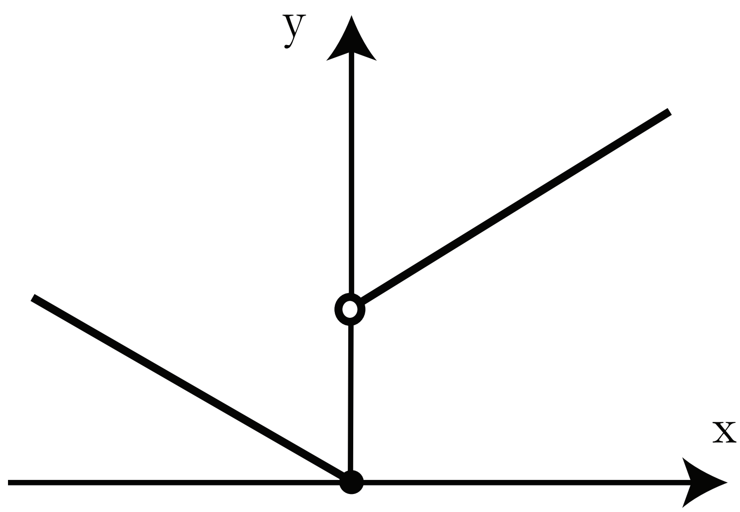

Figure 1: Graph of $f(x)$

The notions of left- and right- hand limits will make things much easier for us as we discuss continuity, next.

Let's talk more about the example graphed above. To calculate

$$\lim_{x \to x_0^+} f(x)$$

we use only values of x that are greater than 0. When $x > 0$, $f(x)$ is defined to equal $x + 1$. So we plugged $x = 0$ into the expression $x + 1$ to calculate the right-hand limit.

When calculating

$$\lim_{x \to x_0^-} f(x),$$

we have $x < 0$. Here $f(x)$ is defined to equal $-x$; when we plug $x = 0$ into this expression we get $\lim\limits_{x \to x_0^-} f(x) = 0$.

Notice that it doesn't matter that $f(0) = 0$. Our calculations would have been exactly the same if $f(0)$ were 1 or even if $f(0) = 2$.