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PROFESSOR: Today, I'm going to continue the idea of setting up integrals. And what we'll deal with is volumes by slices. By slicing. And it's lucky that this is after lunch. Maybe it's after breakfast for some of you, because there's the typical way of introducing this subject is with a food analogy. There's a lot of ways of slicing up food. And we'll give a few more examples than just this one. But, suppose you have, well, suppose you have a loaf of bread here. So here's our loaf of bread, and I hope that looks a little bit like a loaf of bread. It's supposed to be sitting on the kitchen counter ready to be eaten. And in order to figure out how much bread there is there, one way of doing it is to cut it into slices. Now, you probably know that bread is often sliced like this. There are even machines to do it. And with this setup here, I'll draw the slice with a little bit of a more colorful decoration. So here's our red slice of bread. It's coming around like this. And it comes back down behind. So here's our bread slice. And what I'd like to figure out is its volume.

So first of all, there's the thickness of the bread. Which is this dimension, the thickness is this dimension dx here. And the only other dimension that I'm going to give, because this is a very qualitative analysis for now, is what I'll call the area. And that's the area on the face of the slice. And so the area of one slice, which I'll denote by ΔV , that's a chunk of volume, is approximately the area times the change in x . And in the limit, that's going to be something like this. And maybe the areas of the slices vary. There might be a little hole in the middle of the bread somewhere. Maybe it gets a little small on one side. So it might change as x changes. And the whole volume you get by adding up. So if you like, this is one slice. And this is the sum. And you should think of it in a sort of intuitive way as being analogous to the Riemann sum, where you would take each slice individually. And that would look like this. Alright, so that's just a superficial and intuitive way of looking at it.

Now, we're only going to talk about one kind of systematic slice. It's already on your problem set, you had an example of a slice of some region. But we're only going to talk systematically about something called solids of revolution. The idea here is this. Suppose you have some shape, some graph, which maybe looks like this. And now I'm going to revolve it. This is the x -

axis and this is the y-axis. In this case, I'm going to revolve it around the x-axis. If you do that, then the shape that you get is maybe like this. If I can draw it a little bit. It's maybe a football. So that's the shape that you get if you take this piece of disk and you revolve it around. If you had made this copy underneath, it still would have been the same region. So we only pay attention to what's above the axis here. So that's the basic idea.

Now, I'm going to apply the method of slices to figure out the volume of such regions and give you a general formula. And then apply it in a specific case. I want to take one little slice of this football, maybe a football wouldn't work too well, maybe we should go back to a loaf of bread. Anyway, the key point is that you never really have to draw a 3-D picture. And 3-D pictures are awful. They're very hard to deal with. And it's hard to visualize with them. And one of the reasons why we're dealing with solids of revolution is that we don't have as many visualization problems. So we're only going to deal with this. And then you have to imagine from the two-dimensional cross-section what the three-dimensional picture looks like. So we'll do a little bit of an exercise with the three-dimensional picture. But ultimately, you should be used to, getting used to, drawing 2-D diagrams always. To depict the three-dimensional situation. Since it's much harder to draw.

The first step is to consider what this slice is over here. And again it's going to have width dx . And we're going to consider what it looks like over on the 3-D picture. So it starts out being more or less like this. But then we're going to sweep it around. We're revolving around the x-axis, so it's spinning around this way. And if you take this and think of it as being on a hinge, which is down on the x-axis, it's going to swing down and swoop around and come back. Swing around. And that traces out something over here. Which I'm going to draw this way. It traces out a disk. So it's hard for me to draw, and I'm not going to try too hard. Maybe I drew it more like a wheel, looking like a wheel. But anyway, it's this little flat disk here. And so the method that I'm describing for figuring out the volume is called the method of disks. This is going to be our first method.

Now I'm going to apply the reasoning that I have up on the previous blackboard here. Namely, I need to get the volume of this chunk. And the way I'm going to get the volume of this chunk is by figuring out its thickness and its area, its cross-sectional area. And that's not too difficult to do. If this height, so this height is what we usually call y . And y is usually a function of x , it's varying. And this particular distance is y . Then the area of the face is easy to calculate. Because it's a circle, or a disk if you like, with this radius. So its area is πy^2 . So that's, if you

like, one of the dimensions. And then the thickness is dx . So the incremental volume is this. So this is the method of disks. And this is the integrand. Now, there's one peculiar thing about this formula. And there are more peculiar things about this formula. But there's one peculiar thing that you should notice immediately. Which is that I'm integrating with respect to x . And I haven't yet told you what y is. Well, that will depend on what function $y = f(x)$ I use. So we have to plug that in eventually. If we're actually going to calculate something, we're going to have to figure that out. There's another very important point which is that in order to get a definite integral, something I haven't mentioned, we're going to have to figure out where we're starting and we're ending the picture. Which is something we dealt with last time in 2-D pictures.

So let's deal with an example. And we'll switch over from a football to a soccer ball. I'm going to take a circle, and we'll say it has radius a . So this is 0 and this is a . And I'm putting it in this particular spot for a reason. You can do this in lots of different ways, but I'm picking this one to make a certain exercise on your homework easier for you. Because I'm doing half of it for you right now. Appreciate it, yeah. I'm sure especially today it's appreciated. So again, the formula has to do with keeping track of these slices here. And we're sweeping things around. So the full region that we're talking about is the volume of the ball of radius a . That's what our goal is, to figure out what the volume of the ball of radius a is. Alright, again as I say, the thing sweeps around. Coming out of the blackboard, spinning around on this x -axis. So the setup is the following. It's always the same. Here's our formula. And we need to figure out what's going on with that formula. And so we need to solve for y as a function of x . And in order to do that, what we're going to do is just write down the equation for the circle. This is the circle. It's centered at $(a, 0)$, so that's its formula. And now there's one nice thing, which is that we really didn't need it to find the formula for y , we only needed to find the formula for y squared.

So let's just solve for y^2 and we won't have to solve a quadratic or anything. Take a square root, that's nice. This is $a^2 - (x^2 - 2ax + a^2)$. The a^2 's cancel. These two terms cancel. And so the formula here is $2ax - x^2$. Alright now, that is what's known as the integrand. Well, except for this factor of π here. And so the answer for the volume is going to be the integral of π times this integrand, $(2ax - x^2) dx$. And that's just the same thing as this.

But now there's also the issue of the limits. Which is a completely separate problem, which we also have to solve. The range of x is, from this leftmost point to the rightmost point. So x varies, starts at 0 , and it goes all the way up to what, what's the top value here. $2a$. And so now I have a completely specified integral. Again, and this was the theme last time, the whole

goal is to get ourselves to a complete formula for something with an integrand and limits. And then we'll be able to calculate. Now, we have clear sailing to the end of the problem. So let's just finish it off. We have the volume is, if I take the antiderivative of that, that's πax^2 , whose derivative is $2ax$, minus $x^3 / 3$. That's the thing whose derivative is $-x^2$. Evaluated at 0 and $2a$. And that is equal to π times, let's see. $2^2 = 4$. So this is $4a^3 - 8a^3 / 3$, right? And so all told, that is, let's see, $(12/3 - 8/3) \pi a^3$. Which is maybe a familiar formula, $4/3 \pi a^3$. So it worked, we got it right.

Let me just point out a couple of other things about this formula. The first one is that from this point of view, we've actually accomplished more than just finding the volume of the ball. We've also found the volume of a bunch of intermediate regions, which I can draw schematically this way. If I chop this thing, and this portion is x here, then the antiderivative here, this region here, which maybe I'll fill in with this region here, which I'm going to call $V(x)$, is the volume of the portion of the sphere. Volume of portion of width x of the ball. And, well, the formula for it is that it's the volume equals $\pi (ax^2 - x^3 / 3)$. That's it. So we've got something which is actually a lot more information. For instance, if you plug in $x = a$, not surprisingly, and this is a good idea to do because it checks that we've actually got a correct formula here. So if you like you can call this a double-check. If you check $V(a)$, this should be the volume of a half-ball. That's halfway. If I go over here and I only go up to a , that's exactly half of the ball. That had better be half, so let's just see. $V(a)$ in this case is π , and then I have $(a^3 - a^3 / 3)$. And that turns out to be π times a total of $2/3 a^3$, which is indeed half.

Now, on your problem set, what you're going to want to look at is this full formula here. Of this chunk. And what it's going to be good for is a real-life problem. That is, a problem that really came up over the summer, and this fall, at a couple of universities near here, where people were trying to figure out a phenomenon which is well-known. Namely, if you have a bunch of particles in a fluid, and maybe the size of these things is 1 micron. That is, the radius is 1 micron. And then you have a bunch of other little particles, which are a lot smaller. Maybe 10 nanometers. Then what happens is that the particles, the big particles, like to hug each other. They like to clump together, they're very nice. Friendly characters. So what's the explanation for this? The explanation is that actually they are not quite as friendly as they might seem. What's really happening is that the little guys are shoving them around. And pushing them together. And they have sharp elbows, the little ones and they're pushing them. Don't like them to be around and they're pushing them together. But there's actually another possibility. Which is that they also will stick to the sides of the container.

So there are two things that actually happen here. And if you want to get a quantitative handle on how much of this happens, it has to do with how much space these things take up. And so the issue is some kind of overlap between a band around one sphere and a band around the other sphere. And this overlap region is what you have to calculate. You have to calculate what's in here. And you can do that using this formula here. It's not even difficult. So this is, if you cut it in half, turns out to be two of these guys. And then you're on your way to figuring out this problem. And the question is, which do they prefer. Do they prefer to touch each other, or do they prefer to touch the wall. Do they all cluster to the wall. So you can actually see this in solutions, what they do. And the question is, to what extent do they prefer one configuration to the other.

So that's a real, live problem, which really comes up. Came up just this year. And frequently comes up. Which is solved by our first calculation. So now I have, I called this Method 1. For solids of revolution, which is called the method of disks. And now I need to tell you about the other standard method. Which is called the method of shells. So this is our second method. I'm going to illustrate this one with a holiday-themed example here. This is supposed to be a witches' cauldron. Whoops, witch, witches, well. Maybe more than one witch will have this cauldron here. So here's this shape. And we're going to figure out how much liquid is in here. I'm going to plot this. Maybe I'll put this down just a bit lower here. And I'm going to make it a parabola. This is $y = x^2$. And I'm going to make the top height be $y = a$.

So here's my situation. And I want to figure out how much liquid is in here. Now, the reason why I presented the problem in this form, of course, is really to get you used to these things. And the first new thing that I want you to get used to is the idea that now we can also revolve around the y-axis, not just the x-axis. So this one is going to be revolved around the y-axis. And that's what happens. If you spin the parabola around, you get this kind of shape, this kind of solid shape here.

Now, I'm going to use the same kind of slicing that I did before. But it's going to look totally different. Namely, I'll draw it in red again. I'm going to take a little slice over here. And now I want to imagine what happens if it gets revolved around the y-axis. This time it's not a disk. Actually, if I revolve this around the other way, it would have had a hole in it. Which is also possible to do. That's practically the same as the method of disks. You'll maybe discuss that in recitation. Anyway, we're going to revolve it around this way. So again, I need to sweep it around, swing it around like this. And I'll draw the shape. It's going to sweep around in a circle

and maybe it'll have a little bit of thickness to it. And this is the thing that people call a shell. This is the so-called shell of the method. I would maybe call it a cylinder, and another way of thinking about it is that you can maybe wrap up a piece of paper. Like this. There it is, there's a cylinder you can see. Very thin, right? Very thin.

Now, the reason why I used the piece of paper as an example of this is that I'm going to have to figure out the volume of this thing. Its thickness, as usual, is equal to dx . Its height is what? Well, actually I have to use the diagram to see that. The top value is a , and the bottom value is what we call y . So the height is equal to, I'm sorry, $a - y$. Now, again this is an incredibly risky thing here. And I've done this before I pointed this out on the very first day of lecture. The letter y represents a lot of different things. And in disguise, when I call this y , I mean $y = x^2$. In other words, the interesting curve. I don't mean $y = a$, which is the other part.

In general, you might think of it as being equal to $y_{\text{(top)}} - y_{\text{(bottom)}}$. So there are, of course, two y 's involved. In disguise. But we have a shorthand, and the sort of uninteresting one we call by its-- we just evaluate immediately, and the interesting one we leave as the symbol y .

Now, the last bit that I have to do, I claim, in order to figure this out, is the circumference. And the reason for that is that if I think of this thing as like this tube, or piece of paper here, in order to figure out how much stuff there is here, all I have to do is unfold it. Its size, the whole quantity of paper here is the same whether it's rolled up like this or whether it's stretched out. So if I unwrap it, it looks like what? Well, it looks like kind of a slab? Right, it looks like just a slab like this. And again, the thickness is dx . The height is $a - y$. And now we can see that the length here is all the way around. It's the circumference. So this is going to be the circumference when I unwrap it. And in order to figure out the circumference, I need to figure out the radius. So the radius is, on this diagram, is right over there. This is the radius. And that distance is x . So this length here is x . And so this circumference is $2\pi x$. And this height is still $a - y$. And the thickness is still dx .

So in total, we're just going to multiply these numbers together to get the total volume. We have, in other words, dV is equal to the product of the $(2\pi x)$ dimension dimension, the $(a - y)$ dimension, and the dx dimension. Incidentally, dimensional analysis is very useful and important in these problems. You can see that there are three lengths being multiplied together. So we'll get a volume in end. Something cubic. And we will be coming back to that, because it's a quite subtle issue sometimes. So here's the formula, and let's simplify it a little bit. We have $2\pi x$ times, remember, first I have to substitute, otherwise I'm never going to be

able to integrate. And then I rewrite that as $2\pi(ax - x^2, \text{ whoops, } x^3) dx$. Better not get that wrong. And now the last little bit here, that I had better be careful about in order to figure out what the total volume is, is the limits.

So the volume is going to be the integral of this quantity $2\pi(ax - x^3) dx$. And now I have to pay attention to what the limits are. Now, here you have to be careful. x is possibly the-- You have to always go back to the 2-D diagram. I went to it immediately, but that's the whole point. Is that everything gets read off from this diagram here. When you take this guy and you sweep it around, you take care of everything that's to the left. So we only have to count what's to the right. We don't have to count anything over here. Because it's taken care of when we sweep around. So the starting place is going to be $x = 0$. That's where we start. And where we end is the farthest, rightmost spot for x . Which is down here. You've got to watch out about where that is. In the y variable, it's up at $y = a$. But in the x variable, we can see that it's what? It's the square root of a . So these limits, this is where you're going to focus all your attention on the integrand and getting it just right. And then you're going to lose your steam and not pay attention to the limits. They're equally important. You've no hope of getting the right answer without getting the limits right.

So this is the integral from 0 to square root of a . Again, that's just because $y = a$ and $y = x^2$ implies $x = \text{square root } a$. That's that upper limit there. And now, we're ready to carry out this, to evaluate this integral. So we get 2-- sorry, we get $2\pi ax$, that's πax^2 . Maybe I'll leave the 2's in there. $2\pi ax^2$ is the antiderivative of this $ax^2 / 2$, and then here $x^4 / 4$, evaluated at 0 and square root of a . And finally, let's see, what is that. That's $2\pi(a^2 / 2 - a^2 / 4)$. Which is a total of $1/4$, right, $2\pi a^2 / 4$. Which is $\pi/2 a^2$. Yes, question.

STUDENT: [INAUDIBLE]

PROFESSOR: Right. So the question is, why did I integrate only from the middle to this end, instead of all the way from over here, minus square root of a , all the way to the plus end. And the reason is that you have to look at what's happening with the rotation. This red guy, when I swept it around, I counted the stuff to the right and the left. So in other words, if I just rotate the right half of this, I'm covering the left half. So if I counted the stuff from minus square root of a , I would be doing it twice. I would be doubling what I need. So it's too much. Another way of saying it is if I wanted to take the whole region, if I rotate it around only 180 degrees, only π , that would fill up the whole region, if I did both halves. And then instead of a circumference, instead of a $2\pi x$, I could use a πx . But then I would have double what I had. So there are two ways of looking

at it. The same goes, actually, for the football case. When I have that football, I didn't count the bottom part. Because when I swung it around the x-axis, the top part sufficed and I could ignore the bottom half. Yeah, another question.

STUDENT: [INAUDIBLE]

PROFESSOR: Ooh, good question. The question is, when do you know, how do you know when to take the rectangle to be vertical or horizontal. So far we've only done vertical rectangles. And I'm going to do a horizontal example in a second. And the answer to the question of when you do it is this. You can always set it up both ways. One way may be a difficult calculation and one way may be an easier calculation. Yesterday, we did it - or, sorry, the last time. Yeah, I guess it was yesterday. We did it with-- and the horizontal and the vertical were quite different in character. One of them was really a mess, and one of them was a little easier. So very often, one will be easier than the other. Every once in a while, one of them is impossible and the other one is possible. In other words, the difference in difficulty can be extreme. So you don't know that in advance. Yeah, another question.

STUDENT: [INAUDIBLE]

PROFESSOR: The question is, did we just find the volume when you rotate this green region around. Or, did we find the volume when we rotate this whole region. In other words, just the right half or the right and the left half. The answer is both. The region that you get is the same. You always get this cauldron, whether you take this right half when you rotate it around or you take both and you rotate it around. So the answer to both of those questions is the same and it's this. Yes.

STUDENT: [INAUDIBLE]

PROFESSOR: So that if you rotated them both around and you only wanted to cover things once, you would rotate halfway around. Only by 180 degrees. That's true. But you can rotate around as many times as you want. You're still covering the same thing. Over and over and over and over again. So let's go on. I have a very subtle point that I need to discuss with you in order to go on to the next application.

So here's my, and I do want to get to the point of horizontal cross-sections as well. So let's continue here. So the first thing that I want to point out to you now is, I want you to beware of units. There's something a little fishy in this problem. And it can be summarized in the character of the answer, which is just a little bit not clear. Namely, it looks like it's a^2 , right?

And we know that a is in units and we should've gotten cubic units. So there's something a little bit tricky about this question. And so I want to illustrate the paradox right now. So, suppose that $a = 100$ centimeters. And suppose the units are centimeters. Then the formula for the volume is $\frac{\pi}{2} 100^2$. and the units we must take are centimeters cubed. Despite the fact that you kind of want to square the centimeters. But that's not what this problem says. OK, so this is the situation that we've got. Now, if you work out what this is, to figure out what the volume of this cauldron is, what you find is that it's $\frac{\pi}{2}$ times, well, 10^4 is $10 * 1,000$ centimeters cubed. And those are otherwise known as liters. So this is approximately $10 \frac{\pi}{2}$ liters, which is about 16 liters. And so that's how much was in the cauldron under this choice of units.

Now, I'm going to make another choice of units now. And we're going to make a comparison. Suppose that the units are 1 meter. Looks like it should be the same, but if I calculate the volume, it's going to be $\frac{\pi}{2}$, and 1^2 times meters cubed. That's what the formula tells us to do. And if you calculate that out, that's $\frac{\pi}{2} (100 \text{ cm})^3$. And with this unit notation we really do want to cube the centimeters and cube the 100. So we get $\frac{\pi}{2}$. And here we get 10^6 cm^3 . And that comes out to $\frac{\pi}{2} * 1000$ liters. Or, in other words, about 1600 liters. So I'd like to ask you first to contemplate this. And this is a paradox. And this is a serious paradox. If you really want to apply problems, you actually have to understand what your answers mean. So what do you think is going on here? Yeah.

STUDENT: [INAUDIBLE]

PROFESSOR: Yes.

STUDENT: [INAUDIBLE]

PROFESSOR: Right. So the question is, how could either of these make sense. How am I dealing with the units in either case. So now I'm going to explain to you the answer. Because this is really quite puzzling. But it has a resolution. The answer to this question is that both answers are correct. This is correct reasoning in both cases. What's the matter is that you have to interpret the equation $y = x^2$ in two different ways. Two ways. So let me explain what they are. For instance, you can take $y = x^2$ in centimeters. So $y = x^2$ in centimeters. In which case, the picture looks like the following. $a = 100$ centimeters. And this distance here, which is the x value, this is 10. This is 10 centimeters. And that's what the relationship means. So the top of the cauldron, if you like, this distance here is 20 centimeters. This is actually very badly drawn

to scale. It's actually very, very, deep, this thing. It's a rather skinny, deep one. So this is very much not to scale, this picture.

The other picture, the other picture is interpreting $y = x^2$ in meters. And that's more like what I had in mind, actually. I had in mind this big vat here. And this distance here is 1 meter, and then the square root of 1 is 1. So this distance here is also 1 meter. And the top is 2 meters. Now, it's not that crazy. And in fact it's easy to check that, it's pretty reasonable in terms of scale. That this thing has 16 liters in it. And this guy has 1600 liters in it. So you actually have to know what your symbols mean when you're dealing with these kinds of applied problems. And if you're ever really going to do some real consequences, you have to know what the units mean. And the problem with the equation $y = x^2$ is that it's the one that violated scaling rules. Yeah.

STUDENT: [INAUDIBLE]

PROFESSOR: Yeah. **STUDENT:** [INAUDIBLE]

PROFESSOR: No.

STUDENT: [INAUDIBLE]

PROFESSOR: OK, so the question is whether the formula $V = \pi / 2 a^2$. This is the correct answer to the problem. But it is not consistent in units. If you plug in a equals some number of centimeters, some number of millimeters, some number of inches and so on, every single time you'll get a different answer. And they're all inconsistent. In other words, this formula violates scaling.

STUDENT: [INAUDIBLE]

PROFESSOR: If you study each step correctly, you will discover that these are the consistent and correct statements, what I'm writing on the blackboard. And this makes sense in a unit-less sense. But then if you actually stick units on them, one of them, they both are correct. And one of them describes this situation and one of them describes this situation. And it's a mistake to think of this as being 1 meter and cubing the meters. That will be an error that will cause you problems. This 1 is just unit-less, and then the meters cubed got converted. So I encourage you to study this on your own.

So now I'm going to introduce the next problem. We'll have to solve it next time. But the reason why I spent all the time on units is that otherwise it would be impossible for you to

believe me when I do this next calculation. Because we're trying to get a real answer out of a real question. And I'm going to make conversions between centimeters and meters back and forth. And we have to get it consistent in order to have the right answer. So there was a reason for illustrating this pitfall. So this second, the next thing that I'd like to do, is I'd like to boil the water in the witches' cauldron. This is definitely seasonally appropriate, since we're approaching Halloween. And we'll work it out fully next time.

Now, I'm going to introduce another feature into the problem. And this is the one that I want you to understand now. We'll set it all up tomorrow, but right now I need you to understand what the new main idea that we're going to get. There is the new physical feature that I'm going to add to this problem is that if when you're boiling, when the witches are boiling this water, the temperature of the water is not the same at each level in the kettle. At the bottom of the kettle, where you're heating it up, it's at its highest temperature. So at the bottom it's going to be, say, 100 degrees. That is, it's going to be totally boiling. 100 degrees Celsius. And at the top, it's going to be, say, 70 degrees. Right, it's very cold outside. In fact, it's 0 degrees outside. Which is the temperature at which all witches operate, I think. Anyway, so they're boiling their stuff.

And the question that we're going to ask is how much heat, how much heat, do they need to do it. Now, the thing starts out at 0 degrees Celsius. And we're going to heat it up to this temperature configuration here. But it's rising from 100 down here to 70 up here. So the temperature is varying in height. And for simplicity I'm going to make the formula for the temperature be 70 at the top and 100 at the bottom. So it's going to be $100 - 30y$. We'll let the level be 1, here. Sorry, 30. I said 3-- I wrote 3, but I meant 30. So this is the situation that we have.

Now, the point about this problem is we're going to figure out the total temperature, the total amount of heat that you need to add in order to heat this thing up. That's going to be temperature times volume. But some places will count more than others. These will be hotter, but there's less water down here. This is wider up here, so there's more water up here. So there are various things that are varying in this problem. Now, the only way to set up so that it works is to chop things up horizontally instead of vertically. Because it's on the horizontal levels that the temperature is constant. So we'll have an easy calculation for how much it takes to heat up a layer, a horizontal layer. When we rotate this guy around the y axis, that is which kind of shape. It's a disk. So actually this one's going to be an easier problem. It's going to be

a disk problem, not a shell problem. But we're going to have to work things out with respect to the dy variation. In other words, the integral will be with respect to dy . So we will do that next time. We'll figure out how much heat it takes to boil the kettle.