18.01 Single Variable Calculus
Fall 2006

18.01 Solutions to practice questions (Exam 1)

1 a) $D\frac{3t}{\ln t} = \frac{3\ln t - 3t\cdot\frac{1}{t}}{\ln^2 t} = \frac{3(\ln t - 1)}{\ln^2 t}$

When $t = e^2$, $\ln t = 2\ln e = 2$: $= \frac{3(2-1)}{4} = \frac{3}{4}$

b) $\frac{3u}{\tan 2u} = \frac{3u\cdot\cos 2u}{\sin 2u} = \frac{3u\cos 2u}{2\sin u\cos u}$

$= \frac{3}{2}\cdot\frac{u}{\sin u}\cdot\frac{\cos 2u}{\cos u} \to \frac{3}{2}\cdot 1\cdot 1$

c) $D\sin kx = k\cos kx$

$D^2 \cdots = k^2(-\sin kx)$

$D^3 \sin kx = k^3(-\cos kx)$

d) $\frac{d}{d\theta}(a+k\sin^2\theta)^{1/3} = \frac{1}{3}(a+k\sin^2\theta)^{-2/3}\cdot 2k\sin\theta\cos\theta$

$= \frac{2}{3}\frac{k\sin\theta\cos\theta}{(a+k\sin^2\theta)^{2/3}}$

2 $\left.\frac{d}{dx}x^3\right|_{x_0} = \lim_{\Delta x\to 0}\frac{(x_0+\Delta x)^3 - x_0^3}{\Delta x}$

$= \lim_{\Delta x\to 0}\frac{(\cancel{x_0^3}+3x_0^2\Delta x+3x_0(\Delta x)^2+(\Delta x)^3)}{\Delta x}$

$= 3x_0^2$

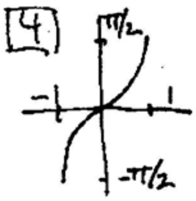
3 On the one hand, $\left.\frac{d}{dx}\sqrt[3]{x}\right|_{x=1} = \left.\frac{1}{3}x^{-2/3}\right|_1 = \frac{1}{3}$

And: by def'n

$\frac{d}{dx}\sqrt[3]{x}\Big| = \lim_{\Delta x\to 0}\frac{\sqrt[3]{1+\Delta x}-\sqrt[3]{1}}{\Delta x} = \lim_{h\to 0}\frac{\sqrt[3]{1+h}-1}{h}$

The correct answer is -1/3.
See details on next page.

$= 1/3$ therefore

4 

$y = \sin^{-1}x$

$y = \sin^{-1}x \iff \sin y = x$

implicit diff'n:

$\cos y\cdot y' = 1$

$y' = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}$

(since $\cos y > 0$ for $-\pi/2 \le y < \pi/2$, choose the positive square root.

5 $f(x) = \begin{cases} ax+b, & x>0 \\ 1-x+x^2, & x\le 0\end{cases}$

Continuous $\iff \left.ax+b\right|_0 = \left.1-x+x^2\right|_0$, or $\boxed{b=1}$

Diff. $\iff$ continuous and slopes are $=$ at 0:

$a = \left.-1+2x\right|_0$ or $\boxed{a=-1}$

6 $x^2y + y^3 + x^2 = 8$ (*)

By implicit diff'n:

$2xy + x^2y' + 3y^2y' + 2x = 0$

Slope horizontal $\iff y' = 0$

$\iff 2x(y+1) = 0$

$\iff x = 0$ or $y = -1$

$x = 0 \underset{\text{by (*)}}{\Rightarrow} y^3 = 8 \Rightarrow y = 2 \quad (0,2)$

$y = -1 \Rightarrow -x^2 - 1 + x^2 = 8 \Rightarrow -1 = 8$ impossible

$\boxed{\therefore \text{ at } (0,2)}$

7 tan line at (x_0, y_0):

$y - y_0 = f'(x_0)(x - x_0)$ At x-intercept, $y=0$

$\therefore -y_0 = f'(x_0)(x-x_0)$

$\therefore x = x_0 - \frac{y_0}{f'(x_0)}$, if $f'(x_0)\ne 0$

8 $V = \frac{4}{3}\pi r^3$ vol of sphere

$\frac{dV}{dt} = 4\pi r^2\cdot\frac{dr}{dt}$, by chain rule

$-10 = 4\pi(20)^2\frac{dr}{dt}$

$\frac{dr}{dt} = \frac{-10}{4\pi(20)^2} = -\frac{1}{160\pi}$ cm/sec

9 a) $\sec x = \frac{1}{\cos x}$; $\cos x = 0$ at $x = \pm\pi/2, \pm 3\pi/2, \ldots$

i.e., at $x = \frac{(2n+1)\pi}{2}$, $n = 0, \pm 1, \ldots$

b) $\frac{1+x^2}{1-x^2} = \frac{1+x^2}{(1-x)(1+x)}$ $\therefore$ at $x=1$, $x=-1$

c)  no slope at $x=0$

10 a) $A = A_0e^{-rt}$

$\frac{A}{A_0} = \frac{1}{4} = e^{-rt}$ take logs:

$-\ln 4 = -rt$

$\therefore t = \frac{\ln 4}{r}$

b) $\frac{dA}{dt} = A_0e^{-rt}\cdot(-r)$

$= A\cdot(-r)$

$= \frac{1}{4}(-r) = -\frac{r}{4}$

Correction to #3

$$\lim_{x\to 0}\left(\frac{1-\sqrt[3]{1+x}}{x}\right)$$

Apply L'Hopital's Rule

$$=\lim_{x\to 0}\left(\frac{-\frac{1}{3(x+1)^{\frac{2}{3}}}}{1}\right)$$

Refine

$$=\lim_{x\to 0}\left(\frac{-1}{3(x+1)^{\frac{2}{3}}}\right)$$

Plug in the value $x=0$

$$=\frac{-1}{3(0+1)^{\frac{2}{3}}}$$

Simplify

$$=-\frac{1}{3}$$