18.01 Single Variable Calculus
Fall 2006

18.01 Problem Set 3

Due Friday 10/06/06, 1:55 pm

Part I   (10 points)

Lecture 11. Fri. Sept. 29. Maximum-minimum problems.
     Read: 4.3, 4.4   Work: 2C-1, 2, 5, 11, 13.

Lecture 12. Tue. Oct. 3   Related rate problems.
     Read: 4.5   Work: 2E-2, 3, 5, 7

Lecture 13. Thu. Oct. 5   Newton's method.
     Read: 4.6, (4.7 is optional)

Lecture 14. Fri. Oct. 6   Mean-value theorem. Inequalities.
     Read: 2.6 to middle p. 77, Notes MVT   Work: assigned on PS4

Part II   (31 points + 8 extra credit)

Directions: Attempt to solve *each part* of each problem yourself. If you collaborate, solutions must be written up independently. It is illegal to consult materials from previous semesters. With each problem is the day it can be done.

0. (not until due date; 3 pts) Write the names of all the people you consulted or with whom you collaborated and the resources you used, or say "none" or "no consultation". (See full explanation on PS1).

1. (Friday, 6pts: 3 + 3)

a) 4.3/28   (Use as variable the distance x from the foot of the ladder to the house. Check endpoints.)

b) 4.4/28

2. (Tuesday, 2pts) **Hypocycloid.** Show that every tangent line to the curve $x^{2/3} + y^{2/3} = 1$ in first quadrant has the property that portion of the line in the first quadrant has length 1. (Use implicit differentiation; this is the same as problem 45 page 114 of text.)

3. (Tuesday, 7pts: 3 + 3 + 1) **Sensitivity of measurement, revisited.**

a) Recall that in problem 2, PS1/Part II, $L^2 + 20,000^2 = h^2$. Use implicit differentiation to calculate dL/dh. Compare the linear approximation dL/dh to the error $\Delta L/\Delta h$ computed in examples on PS1. Explain why $\Delta L/\Delta h \leq dL/dh$ if the derivative is evaluated at the left endpoint of the interval of uncertainty (or, in other words, $\Delta h > 0$). In what range of values of h is it true that $|\Delta L| \leq 2|\Delta h|$?

b) Suppose that the Planet Quirk is a not only flat, but one-dimensional (a straight line). There are several satellites at height $20,000$ kilometers and you get readings saying that satellite 1 is directly above the point $x_1 \pm 10^{-10}$ and is at a distance $h_1 = 21,000 \pm 10^{-2}$ from you, satellite

2 is directly above $x_2 \pm 10^{-10}$ and at a distance $h_2 = 52,000 \pm 10^{-2}$. Where are you and to what accuracy? Hint: Consider separately the cases $x_1 < x_2$ and $x_2 > x_1$.

c) Express dL/dh in terms of the angle between the line of sight to the satellite and the horizontal from the person on the ground. (When expressed using the line-of-sight angle, the formula also works for a curved planet like Earth.)

4. (Tuesday, 5pts: $3 + 2 + 0$) **More sensitivity of measurement.** Consider a parabolic mirror with equation $y = -1/4 + x^2$ and focus at the origin. (See Problem Set 1.) A ray of light traveling down vertically along the line $x = a$ hits the mirror at the point (a, b) where $b = -1/4 + a^2$ and goes to the origin along a ray at angle θ measured from the positive x-axis.

a) Find the formula for $\tan\theta$ in terms of a and b, and calculate $d\theta/da$ using implicit differentiation. (Express your answer in terms of a and θ.)

b) If the telescope records a star at $\theta = -\pi/6$ and the measurement is accurate to 10^{-3} radians, use part (a) to give an estimate as to the location of the star in the variable a.

c) (optional; no credit) Solve for a as a function of θ alone and doublecheck your answers to parts (a) and (b).

5. (Thursday, 8 pts: $3 + 3 + 2$) **Newton's method.**

a) Compute the cube root of 9 to 6 significant figures using Newton's method. Give the general formula, and list numerical values, starting with $x_0 = 2$. At what iteration k does the method surpass the accuracy of your calculator or computer? (Display your answers to the accuracy of your calculator or computer.)

b) For each step x_k, $k = 0, 1, \ldots$, say whether the value is i) larger or smaller than $9^{1/3}$; ii) larger or smaller than the preceding value x_{k-1}. Illustrate on the graph of $x^3 - 9$ why this is so.

c) Find a quadratic approximation to $9^{1/3}$, and estimate the difference between the quadratic approximation and the exact answer. (Hint: To get a reasonable quadratic approximation, use $9 = 8(1 + 1/8)$.)

6. (extra credit 8 pts: $3 + 2 + 2 + 1$) **Hypocycloid, again.** Here we derive the equation for the hypocycloid of Problem 2 from the sweeping out property directly. This takes quite a bit longer. We will look at the hypocycloid from yet another (easier) point of view later on.

Think of the first quadrant of the xy-plane as representing the region to the right of a wall with the ground as the positive x-axis and the wall as the positive y-axis. A unit length ladder is placed vertically against the wall. The bottom of the ladder is at $x = 0$ and slides to the right along the x-axis until the ladder is horizontal. At the same time, the top of the ladder is dragged down the y-axis ending at the origin $(0, 0)$. We are going to describe the region swept out by this motion, in other words, the blurry region formed in a photograph of the motion if the eye of the camera is open the whole time.

a) Suppose that L_1 is the line segment from $(0, y_1)$ to $(x_1, 0)$ and L_2 is the line segment from $(0, y_2)$ to $(x_2, 0)$. Find the formula for the point of intersection (x_3, y_3) of the two line segments. Don't expect the formula to be simple: It must involve all four parameters x_1, x_2, y_1, and y_2. But simplify as much as possible!

It's important to make sure you have the right formulas before proceeding further. You can doublecheck your formulas in several ways. (This is optional.)

i) If $y_2 = 0$, then $x_3 = x_1$.

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ii) When the x's and y's are interchanged the formulas should be the same. What transformation of the plane does the exchange of x and y represent?

iii) It is impossible to find x_3 and y_3 if the lines are parallel, so the denominator in the formula must be zero when L_1 and L_2 have the same slope.

iv) Rescaling all variables by a factor c leaves the formula unchanged, so the numerator of the formula for x_3 and y_3 should have degree (in all variables) one greater than the denominator.

b) Write the equation involving x_2 and y_2 that expresses the property that ladder L_2 has length one. We will suppose that L_1 represents the ladder at a fixed position, and L_2 tends to L_1. Thus

$$x_2 = x_1 + \Delta x; \quad y_2 = y_1 + \Delta y$$

Use implicit differentiation (related rates) to find

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

(Express the limit as a function of the fixed values x_1 and y_1.)

c) Substitute $x_2 = x_1 + \Delta x$ and $y_2 = y_1 + \Delta y$ into the formula in part (a) for x_3 and use part (b) to compute

$$X = \lim_{x_2 \to x_1} x_3 = \lim_{\Delta x \to 0} x_3$$

Simplify as much as possible. Deduce, by symmetry alone, the formula for

$$Y = \lim_{x_2 \to x_1} y_3$$

d) Show that $X^{2/3} + Y^{2/3} = 1$. (The limit point (X, Y) that you found in part (c) is expressed as a function of x_1 and y_1. This is the unique point of the ladder L_1 that is also part of the boundary curve of the region swept out by the family of ladders.)