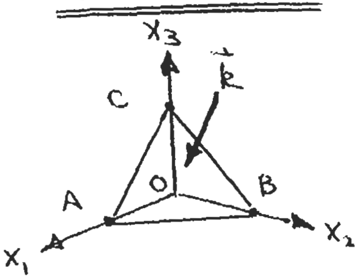
3.60 Symmetry, Structure and Tensor Properties of Materials

THE STRESS TENSOR



CONSIDER A FORCE-DENSITY VECTOR $\vec{k}$ (FORCE PER UNIT AREA) ACTING ON THE EXTERNAL SURFACE ABC OF A VOLUME ELEMENT OABC. WE ASSUME THIS VOLUME ELEMENT IS IN EQUILIBRIUM (NO LINEAR OR ANGULAR ACCELERATION) THE EXTERNAL FORCE DENSITY $\vec{k}$ THAT PUSHES ON THE SOLID MUST, THEREFORE, BE BALANCED BY THE MATERIAL INSIDE THE VOLUME ELEMENT PUSHING BACK AGAINST THE FORCE.

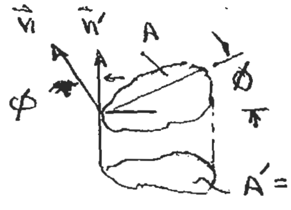
LET US SET UP RELATIONS THAT BALANCE EXTERNAL AND INTERNAL FORCES IN THE X_1 X_2 X_3 DIRECTIONS AS $\vec{k} = k_1, k_2\ k_3$ IS A FORCE DENSITY, WE MUST MULTIPLY BY AN AREA TO GET FORCE.

ALONG X_1 $\quad k_{\textcircled{1}} \cdot \text{AREA } ABC = \sigma_{11} \cdot \text{AREA } BOC + \sigma_{\textcircled{1}2} \cdot \text{AREA } AOC + \sigma_{13} \cdot \text{AREA } AOB$

$\leftarrow X_2$ IS ⊥ AOC

WE HAVE PUT IN QUANTITIES σ, FORCES PER UNIT AREA, WITH TWO SUBSCRIPTS. THE FIRST IS THE DIRECTION IN WHICH THE FORCE DENSITY ACTS, THE SECOND SPECIFIES THE NORMAL TO THE INTERNAL SURFACE ON WHICH THE FORCE DENSITY ACTS. THUS σ_{ij} IS THE FORCE PER UNIT AREA ACTING ALONG X_i ON AN INTERNAL SURFACE WHOSE NORMAL IS X_j

LET US DIVIDE THROUGH BY AREA ABC



$A' = A \cos\phi$

$$k_1 = \sigma_{11} \frac{\text{AREA } BOC}{\text{AREA } ABC} + \sigma_{12} \frac{\text{AREA } AOC}{\text{AREA } ABC} + \sigma_{13} \frac{\text{AREA } AOB}{\text{AREA } ABC}$$

IF WE HAVE AN AREA A IN ONE PLANE AND PROJECT IT ONTO A SURFACE IN A PLANE THAT MAKES AN ANGLE ϕ WITH RESPECT TO THE FIRST, THE NEW AREA $A' = A \cos\phi$

NOTE THAT ϕ IS THE SAME THING AS THE ANGLE BETWEEN THE NORMAL TO THE ORIGINAL SURFACE, $\vec{n}$ AND THE NORMAL TO THE NEW SURFACE $\vec{n}'$. LOOKING AT OUR EQUATION FOR k_1, ABOVE

$\frac{\text{AREA } BOC}{\text{AREA } ABC}$ = COS OF ANGLE BETWEEN THE NORMAL TO THE SURFACE ABC AND THE NORMAL TO BOC, WHICH IS X_1 — AND, THE COSINE OF THE ANGLE BETWEEN ANY VECTOR AND X_1 IS DEFINED AS THE DIRECTION COSINE OF THAT VECTOR

$\therefore \frac{\text{AREA } BOC}{\text{AREA } ABC} = l_1$ A DIRECTION COSINE OF THE NORMAL TO THE SURFACE ABC

SIMILARLY $\quad \frac{\text{AREA } AOC}{\text{AREA } ABC} = l_2 \qquad \frac{\text{AREA } AOB}{\text{AREA } ABC} = l_3$

IT FOLLOWS THAT THE REMAINING EQUATIONS, BALANCING FORCES IN THE X_2 AND X_3 DIRECTIONS ARE

$$\boxed{\begin{array}{l} \left\{ \begin{array}{l} k_1 = \sigma_{11} l_1 + \sigma_{12} l_2 + \sigma_{13} l_3 \\ k_2 = \sigma_{21} l_1 + \sigma_{22} l_2 + \sigma_{23} l_3 \\ k_3 = \sigma_{31} l_1 + \sigma_{32} l_2 + \sigma_{33} l_3 \end{array} \right. \\ k_i = \sigma_{ij} l_j \end{array}}$$

THE NINE COEFFICIENTS σ_{ij} ARE CALLED THE ELEMENTS OF STRESS

$\vec{k} = k_1, k_2, k_3$ IS A VECTOR. WE KNOW THE LAW FOR TRANSFORMATION OF A VECTOR UPON CHANGE OF COORDINATE SYSTEM SPECIFIED BY $X_i' = C_{ij} X_j$, NAMELY $k_i' = C_{ij} k_j$

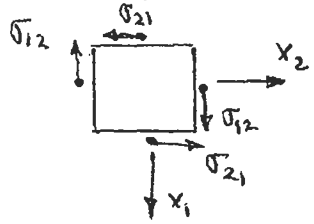
THE ELEMENTS l_i CONSTITUTE A VECTOR — THIS IS $\vec{n}$ A UNIT VECTOR NORMAL TO THE EXTERNAL SURFACE ABC. THE COMPONENTS $n_i' = l_i'$ THUS BECOME $l_i' = C_{ij} l_j$

<u>THE COEFFICIENTS σ_{ij} MUST TRANSFORM LIKE A TENSOR AND, ACCORDINGLY, σ_{ij} IS A TENSOR</u>

The sign of an element of stress can be + or −, we define the following convention: a positive element of stress σ_{ij} is when the material outside the volume element acts on the material inside the volume element to produce a force per unit area in the $+X_i$ direction on a surface whose normal is $+X_j$ (or $-X_i$ and $-X_j$)

If the solid is in equilibrium the stress tensor must be symmetric

Consider the elements of stress σ_{12} and σ_{21}



σ_{12} and the same σ_{12} on the opposite side of the volume element create a couple that would act to produce an angular acceleration. σ_{21} is the only couple that can balance σ_{12}.

$\therefore\ \sigma_{12} = \sigma_{21}$ and $\boxed{\sigma_{ij} = \sigma_{ji}}$ for equilibrium.

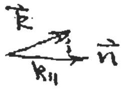
As stress is a second-rank tensor, we can define a stress quadric

$$\boxed{\sigma_{ij} x_i x_j = 1}$$

Everything that we have said about the representation quadric applies to the stress quadric.

(a) Reference axes can be changed $X_1 X_2 X_3 \rightarrow X_1' X_2' X_3'$ to place the tensor in a diagonal form

(b) Using $\sigma_{ij} \ell_i \ell_j = \sigma$ we can compute the "value of stress in a particular direction". We'll have to think a moment to see what that means! The "applied vector" in $k_i = \sigma_{ij} \ell_j$ is the set of direction cosines of the normal to the surface. The "generalized displacement" is $\vec{k}$ the imposed force density.



What $\sigma_{ij} \ell_i \ell_j$ provides is $\dfrac{k_{\parallel}}{\text{unit normal}} = \dfrac{k_{\parallel}}{|\vec{n}|} = k_{\parallel}$ This is the force per unit area that is parallel to the normal to the surface that is transmitted across this surface in other words the tensile component of the transmitted stress.

(c) Some special forms of the stress tensor when in diagonal form

$$\begin{bmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Uniaxial stress

$$\begin{bmatrix} \sigma_{11} & 0 & 0 \\ 0 & \sigma_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Biaxial stress

$$\begin{bmatrix} \sigma_{11} & 0 & 0 \\ 0 & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{bmatrix}$$

Triaxial stress

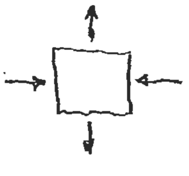
$$\begin{bmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{bmatrix}$$

Hydrostatic pressure

$$\begin{bmatrix} \sigma & 0 & 0 \\ 0 & -\sigma & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[45^\circ]{\text{Rotate axes}} \begin{bmatrix} 0 & \sigma' & 0 \\ \sigma' & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Shear stress

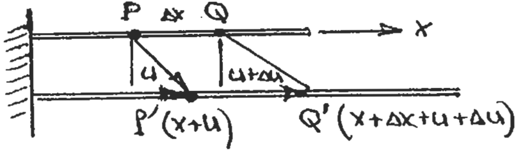
Note: $\sigma' \neq \sigma$



Stress is a field tensor not a property tensor!

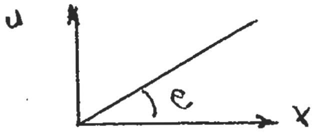
STRAIN

Let us introduce the concept of strain with a one-dimensional example. Let us consider an elastic band attached to a wall.



If we exert a force on the elastic bank, point P at location X before the deformation, undergoes a displacement $\vec{u}$ to point P′ at (X+u). A point Q at distance ΔX from P is also displaced: it moves a distance u+Δu and is thus displaced to Q′(X+ΔX+u+Δu)

If we plot u as a function of distance along the elastic band we find that u



will increase linearly with X. This is termed Homogeneous Deformation the slope of the plot is linear and $\partial u/\partial x = e$

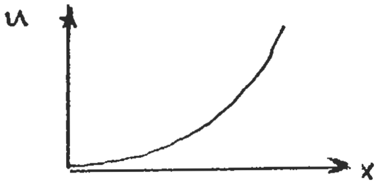
Let's return to our points P & Q. It is not displacement that interest us as much as relative displacement as this causes changes in length and forces within the material

$$\begin{cases} P(X) \text{ has been mapped to } P'(X+u) = P'(X+eX) = P'[X(1+e)] \\ Q(X+\Delta X) \longrightarrow Q'[(X+\Delta X)+(u+\Delta u)] = Q'[(X+\Delta X)+e(X+\Delta X)] = Q'[(X+\Delta X)(1+e)] \end{cases}$$

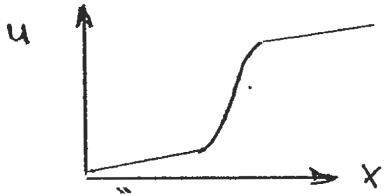
$$\therefore P'Q' = \Delta X(1+e)$$

Relative change of length is $\dfrac{\Delta X(1+e)-\Delta X}{\Delta X} = e = \dfrac{\Delta u}{\Delta X} \equiv$ Strain (Dimensionless)

Before continuing, we should pause to note that the real world displays types of strain in adding to homogeneous strain



Inhomogeneous strain $\partial u/\partial x$ is a function of x



One portion of the sample yields and rapidly elongates relative to other parts

"Necking" – commonly seen in polymers and metals

EXTENSION TO THREE DIMENSIONS

In the above expression for one dimension we started by saying $u = eX$ $\left(\frac{\partial u}{\partial x} = e\right)$ but then showed that changes in length behave the same way $\frac{\Delta u}{\Delta X} = e$ = fractional change in length of line segment.

What we essentially said was that $\vec{u}$ is proportional to position

To express the fractional change of length in different directions in three dimensions let's assume that $\vec{u}$ is proportional to (but not necessarily parallel to) position in the material, $\vec{X}$.

We can then write

$$\begin{cases} u_1 = \dfrac{\partial u_1}{\partial x_1} X_1 + \dfrac{\partial u_1}{\partial x_2} X_2 + \dfrac{\partial u_1}{\partial x_3} X_3 \\ u_2 = \dfrac{\partial u_2}{\partial x_1} X_1 + \dfrac{\partial u_2}{\partial x_2} X_2 + \dfrac{\partial u_2}{\partial x_3} X_3 \\ u_3 = \dfrac{\partial u_3}{\partial x_1} X_1 + \dfrac{\partial u_3}{\partial x_2} X_2 + \dfrac{\partial u_3}{\partial x_3} X_3 \end{cases}$$

or $\boxed{u_i = \partial u_i/\partial x_j \; X_j}$

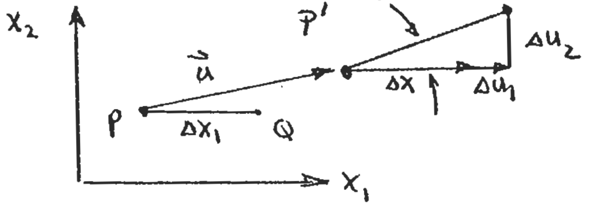
or in terms of differences in displacement

$$\begin{cases} \Delta u_1 = \dfrac{\partial u_1}{\partial x_1} \Delta X_1 + \dfrac{\partial u_1}{\partial x_2} \Delta X_2 + \dfrac{\partial u_1}{\partial x_3} \Delta X_3 \\ \vdots \end{cases}$$

or $\boxed{\Delta u_i = \partial u_i/\partial x_j \; \Delta X_j}$

Let's define $\boxed{\partial u_i/\partial x_j = e_{ij}}$

Meaning of e_{ij}



Let us consider a line segment PQ of length ΔX_1 that is initially parallel to X_1

$$\left\{\begin{aligned} \Delta u_1 &\simeq \frac{\partial u_1}{\partial x_1}\Delta X_1 = e_{11}\,\Delta X_1 \\ \Delta u_2 &= \frac{\partial u_2}{\partial x_1}\Delta X_1 = e_{21}\,\Delta X_1 \end{aligned}\right.$$

↳ Selection of PQ as initially parallel to X_1 means there is no contribution ΔX_2. This keeps things simple

The line segment $PQ = \Delta X_1$ has changed length and has rotated direction.

The change of length is $\{(\Delta X + \Delta u_1)^2 + \Delta u_2^2\}^{\frac{1}{2}} \approx \Delta X + \Delta u_1$ or, more rigorously

It the change of length in $P'Q'$ resolved along X_1 $\boxed{\therefore e_{11} = \frac{\Delta u_1}{\Delta X_1} = \text{Tensile strain along } X_1}$

The line segment $P'Q'$ has also rotated an amount ϕ $\boxed{e_{ii} = \partial u_i/\partial x_i = \text{Tensile strain along } X_i}$

$$\tan\phi = \frac{\Delta u_2}{\Delta X_1 + \Delta u_1} \approx \frac{\Delta u_2}{\Delta X_1} = e_{21} \approx \phi$$ if e_{21} is small an $\tan\phi \approx \phi$

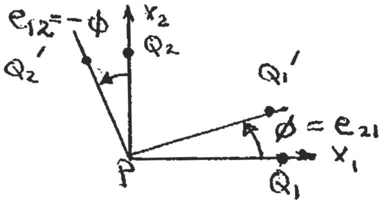
$\therefore e_{21}$ = Angle of rotation of line segment initially along X_1 in the direction of X_2

or, general $\underline{e_{ij} = \partial u_i/\partial x_j \simeq \text{Angle of rotation of line segment along } X_j \text{ in the direction}}$ of axis X_i

Proper Definition of a Satisfactory Measure of Strain

Consider, in a 2-dimensional space, a state of "strain" in which $e_{12} = -e_{21}$ $e_{ii} = 0$

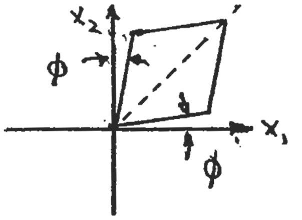
that would produce displacements in which a line PQ_1 initially along X_1 is rotated toward X_2 by an angle $\phi \simeq e_{12}$

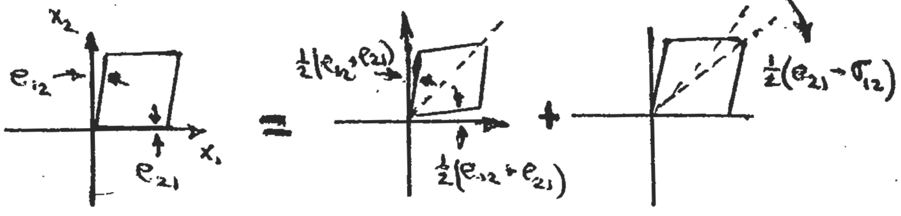


Consider a line PQ_2 initially along X_2. If e_{12} is $-e_{21}$ it rotates $-\phi$ to a new location PQ_2' (If $+e_{ij}$ rotates a line along X_j toward X_i then a negative e_{ij} rotates a line along X_j away from X_i)

Would one call this a "deformation"? No! It looks pretty much like rotation about X_3 with out any strain at all!

Indeed, if you picked up what had been initially a cube of material, and wanted to place it in a position so that angles such as e_{21} and σ_{12} really did provide a measure of deformation, you would probably, by instinct, place the deformed block relative to the reference axes such that $e_{ij} = e_{ji}$. In other words, if $e_{21} \neq e_{12}$, you would rotate the body until it was





THIS IS A NEW CONCEPT : ADDITION of TENSORS !

LET US TAKE THE TENSOR e_{ij} AND DIVIDE IT INTO TWO PARTS

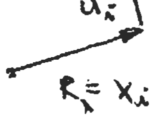
$$e_{ij} = \epsilon_{ij} + \omega_{ij} \quad \text{with} \begin{cases} \epsilon_{ij} \equiv \frac{1}{2}(e_{ij} + e_{ji}) \\ \omega_{ij} \equiv \frac{1}{2}(e_{ij} - e_{ji}) \end{cases}$$

$$e_{ij} = \epsilon_{ij} + \omega_{ij} = \frac{1}{2}(2e_{ij})$$

THE TENSOR ω_{ij} will, from THE WAY IN which IT WAS DEFINED, have $\omega_{ii} = 0$ AND $\omega_{ij} = -\omega_{ji}$. IT IS AN "ANTISYMMETRIC TENSOR"

PROOF THAT PURE ROTATION WITH NO STRAIN CORRESPONDS AN ANTI-SYMMETRIC e_{ij}

If e_{ij} RELATES A DISPLACEMENT VECTOR u_i TO A POSITION IN THE SOLID $\vec{R} = (X_1, X_2, X_3)$ AND e_{ij} CORRESPONDS TO PURE RIGID BODY ROTATION, THEN A CHARACTERISTIC of u_i,



THEN A CHARACTERISTIC of $\vec{u}$ IS THAT IT IS EVERYWHERE $\perp$ $\vec{R}$ REGARDLESS of THE COORDINATES $X_1 X_2 X_3$ TO which $\vec{R}$ EXTENDS

FOR THIS TO BE TRUE $\vec{u} \cdot \vec{R} = 0$ (NOTE: STRAIN IS ASSUMED TO BE SMALL!)

$$\text{OR} \quad u_i R_i = 0 \qquad u_i = e_{ij} X_j$$

$$e_{ij} X_i X_j = 0 \qquad R_i = X_i$$

EXPANDING THE ABOVE EXPRESSION

$$e_{11} X_1^2 + e_{22} X_2^2 + e_{33} X_3^2 + (e_{13} + e_{31}) X_1 X_3 + (e_{23} + e_{32}) X_2 X_3 + (e_{12} + e_{21}) X_1 X_2 = 0$$

FOR THIS TO BE TRUE for ALL X_i, THE COEFFICIENTS of $X_i X_j$ MUST INDIVIDUALLY VANISH!

$$\therefore e_{11} = e_{22} = e_{33} = 0 \qquad \begin{cases} e_{13} + e_{31} = 0 \\ e_{23} + e_{32} = 0 \\ e_{12} + e_{21} = 0 \end{cases} \quad \begin{cases} e_{13} = -e_{31} \\ e_{23} = -e_{32} \\ e_{12} = -e_{21} \end{cases}$$

THEREFORE RIGID BODY ROTATION CORRESPONDS TO

$$\begin{bmatrix} 0 & e_{12} & e_{13} \\ -e_{12} & 0 & e_{23} \\ -e_{31} & -e_{32} & 0 \end{bmatrix} \qquad \text{QED!}$$

To DEFINE TRUE STRAIN ϵ_{ij} — THAT IS, REAL DEFORMATION, WE THEREFORE SPLIT A GENERAL e_{ij} INTO TWO PARTS $e_{ij} = \epsilon_{ij} + \omega_{ij}$ with $\epsilon_{ij} = \frac{1}{2}(e_{ij} + e_{ji})$ $\omega_{ij} = \frac{1}{2}(e_{ij} - e_{ji})$

THE final RESULT IS, for TRUE DEFORMATION

$$u_i = \epsilon_{ij} X_i \qquad u_i \text{ IS A DISPLACEMENT VECTOR, } X_i \text{ IS A POSITION VECTOR.}$$

u_i AND X_i BOTH TRANSFORM ACCORDING TO THE LAW FOR 1ST RANK TENSORS, THEREFORE ϵ_{ij} IS A TENSOR, THE STRAIN TENSOR, of 2ND RANK.

ALL of THE PROPERTIES THAT WE HAVE ASCRIBED TO TENSORS IN PREVIOUS DISCUSSION THEREFORE APPLY TO THE STRAIN TENSOR

- → THE ELEMENTS of STRAIN DEPEND ON THE COORDINATE SYSTEM TO which THE ARE REFERRED. UPON CHANGE of REFERENCE AXES $X_1 X_2 X_3 \rightarrow X_1' X_2' X_3'$ DESCRIBED BY $[C_{ij}]$, $\epsilon_{ij}' = C_{il} C_{jm} \epsilon_{lm}$

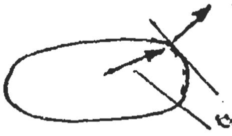
- THE "VALUE OF STRAIN" IN A GIVEN DIRECTION DEFINED BY DIRECTION COSINES l_i IS $\epsilon = \epsilon_{ij} l_i l_j$. WHAT DOES IT MEAN? TURNING TO OUR DEFINITION OF VALUE OF A PROPERTY IN A GIVEN DIRECTION,

$$\text{if } q_i = a_{ij} p_j \qquad a = a_{ij} l_i l_j = \frac{q_{11}}{|p|}$$

IN THE PRESENT CASE $\epsilon = \dfrac{u_{11}}{|R|}$ — IT IS THE COMPONENT OF $\vec{u}$ IN THE DIRECTION OF $\vec{R}$, OR THE TENSILE DISPLACEMENT PER UNIT LENGTH OF R

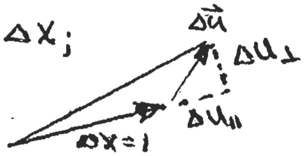
- A STRAIN QUADRIC $\epsilon_{ij} x_i x_j = 1$ CAN BE DEFINED AND HAS ALL OF THE PROPERTIES THAT WE ASCRIBED TO THE REPRESENTATION QUADRIC — INCLUDING THOSE THAT ARE VALID ONLY FOR SYMMETRIC TENSORS (AS WE HAVE DEFINED ϵ_{ij} AS SYMMETRIC)
  - THE STRAIN QUADRIC, ACCORDINGLY, HAS A RADIUS THAT IS $1/\sqrt{\epsilon}$ IN THE DIRECTION OF THE RADIUS

THE "RADIUS NORMAL" PROPERTY WORKS BECAUSE ϵ_{ij} IS SYMMETRIC



UNIT LENGTH
IN A PARTICULAR ORIENTATION
THAT DEFINES THE DIRECTION OF $\vec{R}$

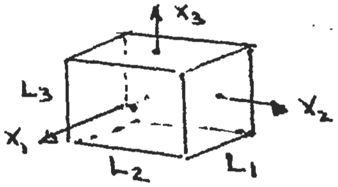
$$\Delta U_i = \epsilon_{ij} \Delta X_j$$



THE INITIAL LINE SEGMENT IS ROTATED AND CHANGES LENGTH $\epsilon_{ij} l_i l_j$ GIVES ϵ_{11} PER UNIT ΔX AND IS CALLED THE STRETCH IN THAT DIRECTION

- A STRAIN TENSOR IN GENERAL FORM CAN BE CONVERTED TO A DIAGONAL FORM BY FINDING THE EIGENVALUES AND EIGENVECTORS AND REFERRING THE TENSOR TO THE NEW PRINCIPAL AXES.

VOLUME CHANGE ASSOCIATED WITH DEFORMATION



CONSIDER A VOLUME ELEMENT, A BOX WITH EDGES L_1, L_2, L_3

TO KEEP THINGS SIMPLE, LET'S ASSUME THAT WE HAVE TAKEN THE REFERENCE AXES ALONG THE PRINCIPAL AXES OF THE STRAIN QUADRIC SO THAT THE STRAIN TENSOR IS DIAGONAL

AFTER DEFORMATION
$$\begin{cases} L_1 \rightarrow L_1(1+\epsilon_{11}) \\ L_2 \rightarrow L_2(1+\epsilon_{22}) \\ L_3 \rightarrow L_3(1+\epsilon_{33}) \end{cases} \qquad \begin{bmatrix} \epsilon_{11} & 0 & 0 \\ 0 & \epsilon_{22} & 0 \\ 0 & 0 & \epsilon_{33} \end{bmatrix}$$

THEN, IF $V = L_1 L_2 L_3$

$$V' = V + \Delta V = L_1(1+\epsilon_{11}) \cdot L_2(1+\epsilon_{22}) \cdot L_3(1+\epsilon_{33})$$
$$= L_1 L_2 L_3 (1 + \epsilon_{11} + \epsilon_{22} + \epsilon_{11}\epsilon_{22})(1+\epsilon_{33})$$
$$= L_1 L_2 L_3 (1 + \epsilon_{11} + \epsilon_{22} + \cancel{\epsilon_{11}\epsilon_{22}} + \cancel{\epsilon_{11}\epsilon_{33}} + \cancel{\epsilon_{22}\epsilon_{33}} + \cancel{\epsilon_{11}\epsilon_{22}\epsilon_{33}})$$

AS STRAINS ARE SMALL, NEGLECT HIGHER-ORDER TERMS

$$\therefore\ V + \Delta V \approx L_1 L_2 L_3 (1 + \epsilon_{11} + \epsilon_{22} + \epsilon_{33})$$
$$V + \Delta V = V + \underbrace{V(\epsilon_{11} + \epsilon_{22} + \epsilon_{33})}_{\Delta V}$$

$$\boxed{\frac{\Delta V}{V} = \epsilon_{11} + \epsilon_{22} + \epsilon_{33}}$$

THIS IS THE FRACTIONAL CHANGE IN VOLUME CAUSED BY DEFORMATION ~ CALLED THE DILATION

We showed that the fractional volume change upon deformation is $\epsilon_{11}+\epsilon_{22}+\epsilon_{33}$. Because $\Delta V/V$ is a scalar quantity, we would <u>not</u> expect the dilation to change upon change of reference axes. Indeed, it does not. This sum of diagonal elements for a second-rank tensor is a generally-defined quantity for tensors and is called the <u>trace</u> of the tensor

$$\boxed{\text{Trace} = T = a_{11}+a_{22}+a_{33}}$$

<u>It has the property that it remains invariant upon any change of reference axes.</u>

(This is straightforward to prove: add together $a_{11}' = c_{1l}c_{1m}a_{lm} +$

$$c_{2l}c_{2m}a_{lm} + c_{3l}c_{3m}a_{lm} = \underbrace{(c_{1l}c_{1m}+c_{2l}c_{2m}+c_{3l}c_{3m})}_{\begin{array}{l} = 0 \ l \neq m \\ = 1 \ l = m \end{array}} a_{lm}$$

$\left.\begin{array}{l} = 0 \quad l \neq m \\ = 1 \quad l = m \end{array}\right\}$ from the orthogonality properties of the direction cosine matrix

$$\therefore T' = a_{11}+a_{22}+a_{33} \quad \text{QED} \;)$$

<u>Some special forms of the diagonalized strain tensor</u>

Plane strain

$$\begin{bmatrix} \epsilon_{11} & 0 & 0 \\ 0 & \epsilon_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

"Pure shear" (special case of plane strain)

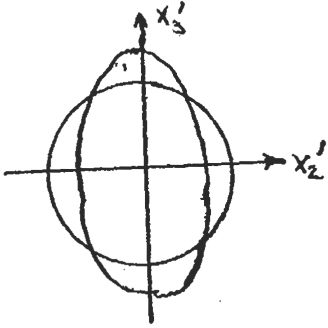
$$\begin{bmatrix} -\epsilon_{11} & 0 & 0 \\ 0 & \epsilon_{11} & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[\text{by } 45^\circ]{\text{rotate axes}} \begin{bmatrix} 0 & \epsilon' & 0 \\ \epsilon' & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Note that trace ≡ 0 shear causes no volume change!

Some other non-tensor definitions involving strain

"Simple shear"

One sometimes sees reference to the "strain ellipsoid"



Consider a specimen initially in the form of a sphere of unit radius

$$x_1^2 + x_2^2 + x_3^2 = 1$$

After deformation (taking x_1' x_2' x_3' along principal axes)

$$\left\{\begin{array}{l} x_1' = x_1(1+\epsilon_{11}) \\ x_2' = x_2(1+\epsilon_{22}) \\ x_3' = x_3(1+\epsilon_{33}) \end{array}\right.$$

The equation of the surface after deformation will be

$$\frac{x_1^2}{(1+\epsilon_{11})^2} + \frac{x_2^2}{(1+\epsilon_{22})^2} + \frac{x_3^2}{(1+\epsilon_{33})^2} = 1$$

This is the "<u>strain ellipsoid</u>" (it is <u>not</u> the strain quadric!!)