Compatibility Equations

Given that $\epsilon_{ij} = \epsilon_{ji}$, the necessary conditions that there exist displacements u_i such that $\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$ are:

$$\boxed{e_{ijk}\, e_{mnp}\, \epsilon_{jn,kp} = 0} \rightarrow \text{gives the 6 compatibility equations}$$

Proof: Assuming u_i exists:

$$u_{i,j} = \underbrace{\tfrac{1}{2}(u_{i,j} + u_{j,i})}_{\substack{\epsilon_{ij} \\ \text{symmetric}}} + \underbrace{\tfrac{1}{2}(u_{i,j} - u_{j,i})}_{\substack{\vartheta_{ij} \\ \text{antisymmetric}}}$$

$\vartheta_{ij} \rightarrow$ small rotation tensor (has no utility if rotations are large)

symmetric: $\epsilon_{ij} = \epsilon_{ji}$ Antisymmetric: $\vartheta_{ij} = -\vartheta_{ji} \rightarrow \vartheta_{ij} = -e_{ijk}\,\omega_k$, where e_{ijk} is the permutation tensor, $\vec{\omega}$ is the small rotation vector

$u_{i,j} = \epsilon_{ij} + \vartheta_{ij} = \epsilon_{ij} - e_{ijk}\,\omega_k$ — ① $\rightarrow$ Take second derivative

$u_{i,jp} = \epsilon_{ij,p} - e_{ijk}\,\omega_{k,p}$ — ② $\rightarrow$ Multiply both sides by e_{rjp}

$e_{rjp}\, u_{i,jp} = e_{rjp}\,\epsilon_{ij,p} - e_{rjp}\, e_{ijk}\,\omega_{k,p}$ — ③

$$\underbrace{e_{rjp}}_{\text{antisymmetric}}\, \underbrace{u_{i,jp}}_{\text{symmetric}} = 0$$

$e_{rjp}\,\epsilon_{ij,p} = e_{rjp}\, e_{ijk}\,\omega_{k,p}$ — ④ $\rightarrow$ Rearrange indices in permutation tensors on RHS

$e_{rjp}\,\epsilon_{ij,p} = e_{prj}\, e_{kij}\,\omega_{k,p} \rightarrow$ Note: $e_{prj}\, e_{kij} = (\delta_{pk}\delta_{ri} - \delta_{pi}\delta_{rk})$

$e_{rjp}\,\epsilon_{ij,p} = (\delta_{pk}\delta_{ri} - \delta_{pi}\delta_{rk})\,\omega_{k,p} \rightarrow$ <u>Note:</u> $\delta_{pk}\,\omega_{k,p} = \omega_{k,k}$; $\delta_{rk}\,\omega_{k,p} = \omega_{r,p}$

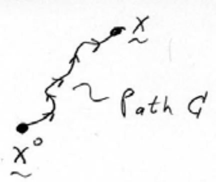
$= \delta_{ri}\,\omega_{k,k} - \delta_{pi}\,\omega_{r,p} \rightarrow$ <u>Note</u>: $\delta_{pi}\,\omega_{r,p} = \omega_{r,i}$

$\omega_k = -\frac{1}{2} e_{kmn}\,\vartheta_{mn} = \frac{1}{2} e_{kmn}\, u_{n,m}$

$\therefore \boxed{\omega_{k,k} = 0}$

$$\boxed{e_{rjp}\,\epsilon_{ij,p} = -\omega_{r,i}} \text{ — ⑤}$$

Path Independence:



$$\omega_r(\underset{\sim}{x}) - \omega_r(\underset{\sim}{x}^\circ) = \oint_{\underset{\sim}{x}^\circ}^{\underset{\sim}{x}} \omega_{r,i}\, dx_i = -\oint_{\underset{\sim}{x}^\circ}^{\underset{\sim}{x}} e_{rjp}\,\epsilon_{ij,p}\, dx_i \text{ — ⑥}$$

Basic theorem for the existence of path independence:

If $A_i = F_{,i} \longrightarrow e_{ijk} A_{k,j}$ for path independence

$\bar{A} = \bar{\nabla} F \qquad (\bar{\nabla} \times \bar{A} = 0)$

Apply this theorem to Equation 5

$$\boxed{e_{min} e_{rjp} \epsilon_{ij,pn} = 0}$$

These are the compatibility equations!!

$$\boxed{\begin{aligned}
&\epsilon_{11,22} + \epsilon_{22,11} - 2\epsilon_{12,12} = 0 \\
&\epsilon_{11,33} + \epsilon_{33,11} - 2\epsilon_{13,13} = 0 \\
&\epsilon_{22,33} + \epsilon_{33,22} - 2\epsilon_{23,23} = 0 \\
&-\epsilon_{23,11} + \epsilon_{13,12} + \epsilon_{12,13} = \epsilon_{11,23} \\
&-\epsilon_{13,22} + \epsilon_{12,23} + \epsilon_{23,12} = \epsilon_{22,13} \\
&-\epsilon_{12,23} + \epsilon_{23,13} + \epsilon_{13,23} = \epsilon_{33,12}
\end{aligned}}$$