3.185 Problem Set 4

Introduction to Heat Transfer

Solutions

1. Thermal properties and optimal materials selection

   (a) The steady flux is given by Fourier's law: $q = -k\frac{dT}{dx}$. If we have a hot body and a cold body at certain temperatures and a certain distance apart, then $\frac{dT}{dx}$ is fixed, and we want to minimize k. Therefore, silica is by far the best choice.

   (b) We want a long timescale, so that the heat bursts are damped by the heat shield. Since the timescale is $\frac{L^2}{\alpha}$, we want to minimize α, and the best material is again silica.

   (c) Here, we want short timescale, so we maximize α (and exclude diamond), giving us silver as the optimal material.

   (d) All we need to do is maximize heat energy per unit weight per degree, and the heat capacity c_p measures just that. So, we maximize c_p, which gives us aluminum as the material of choice. (Note that in these units water has a heat capacity of $4184\frac{\mathrm{J}}{\mathrm{kg \cdot K}}$, which is over four times better.)

   (e) We want to minimize ΔT for a given q. If we solve Fourier's 1st law for ΔT, we find it is equal to $\frac{q\Delta x}{k}$. With q and Δx fixed, minimizing ΔT means maximizing k, and the choice is diamond.

   (f) Here, we want to maximize the flux for an unsteady problem. If we look at the erfc solution, which is valid for the time of initial contact between the molten metal and the rotating wheel, we find that $T = T_i + (T_0 - T_i)\mathrm{erfc}\left(\frac{y}{2\sqrt{\alpha t}}\right)$, where y is the distance from the wheel's outer surface. If we evaluate the flux through the surface using $q = -k\frac{dT}{dy}$, this gives us $q = -k(T_0 - T_i)\frac{2}{\sqrt{\pi}}\frac{1}{2\sqrt{\alpha t}}$. The flux is proportional to $\frac{k}{\sqrt{\alpha}}$, which is equal to $\sqrt{k\rho c_p}$. The non-diamond material with the largest value of this parameter is copper. (Perhaps as diamond films fall in price they too will be used in this application.)

   Candidate materials:

Material	k, $\frac{\mathrm{W}}{\mathrm{m \cdot K}}$	ρ, $\frac{\mathrm{g}}{\mathrm{cm}^3}$	c_p, $\frac{\mathrm{J}}{\mathrm{kg \cdot K}}$	α, $\frac{\mathrm{cm}^2}{s}$	$\sqrt{k\rho c_p}$, $\frac{\mathrm{W}\sqrt{\mathrm{s}}}{m^2 \cdot K}$
aluminum	238	2.7	917	0.96	2.43×10^4
copper	397	8.96	386	1.14	3.71×10^4
gold	315.5	19.3	130	1.26	2.81×10^4
silver	425	10.5	234	1.73	3.23×10^4
diamond	2320	3.5	519	12.8	6.49×10^4
graphite	63	2.25	711	0.225	1.00×10^4
lime (CaO)	15.5	3.32	749	0.0623	6.21×10^3
silica (SiO_2)	1.5	2.32	687	0.0094	1.55×10^3
alumina (Al_2O_3)	39	3.96	804	0.122	1.11×10^4

2. Layered furnace wall and British units

   Set R_1 to the inner radius and T_1 to the temperature there, which equals the melt temperature of 2000°F. Set the radius of the graphite-brick interface to R_2, and the temperature there to T_2 (not given). Set the outer radius of the brick later to R_3, the outer brick temperature to T_3 (also not given), and the environment temperature to T_4 (70°F).

   (a) This is a sum of resistances problem, with the linear solution for the top and bottom given in class:

   $$q_z = \frac{T_1 - T_4}{\frac{L_1}{k_1} + \frac{L_2}{k_2} + \frac{1}{h}}$$

   The first material is graphite, L_1 is 1.5 ft and k_1 is 63 $\frac{\text{W}}{\text{m}\cdot\text{K}} \times 0.557 = 35.1\frac{\text{BTU}}{\text{hr}\cdot\text{ft}\cdot{}^\circ\text{F}}$; for brick, $L_2 = 4$ ft and $k_2 = 16\frac{\text{BTU}}{\text{hr}\cdot\text{ft}\cdot{}^\circ\text{F}}$; $h = 4\frac{\text{BTU}}{\text{hr}\cdot\text{ft}^2\cdot{}^\circ\text{F}}$. So this gives us

   $$q_z = \frac{1930^\circ\text{F}}{0.543\frac{\text{hr}\cdot\text{ft}^2\cdot{}^\circ\text{F}}{\text{BTU}}} = 3556\frac{\text{BTU}}{\text{hr}\cdot\text{ft}^2}$$

   The area of the top and bottom are πR^2, or 314 ft^2 each, so twice this times the flux gives $2.23 \times 10^6\frac{\text{BTU}}{\text{hr}}$.
   In the radial direction, the equivalent is written in terms of Q, the flux-area product:

   $$Q = \frac{2\pi L(T_1 - T_4)}{\frac{1}{k_1}\ln\frac{R_2}{R_1} + \frac{1}{k_2}\ln\frac{R_3}{R_2} + \frac{1}{hR_3}}.$$

   With our parameters, this gives us

   $$\frac{1.82 \times 10^5 \text{ft}\cdot{}^\circ\text{F}}{0.0388\frac{\text{hr}\cdot\text{ft}\cdot{}^\circ\text{F}}{\text{BTU}}} = 4.69 \times 10^6\frac{\text{BTU}}{\text{hr}}$$

   Adding the power through the sides to that through the top and bottom gives a total power of $6.92 \times 10^6\frac{\text{BTU}}{\text{hr}}$. You could convert the units using 1 BTU=1055 J, and 1 hr=3600 s, so the total power is 2030 kW (but you didn't have to).

   (b) By ignoring the corners, we treat them as perfect insulators. In a real furnace, they too would conduct heat away, so our power number is an *underestimate*.

   (c) We can just use the equation above with our known Q and a single layer at a time, *e.g.* for the graphite layer:

   $$Q = \frac{2\pi L(T_1 - T_2)}{\frac{1}{k_1}\ln\frac{R_2}{R_1}}$$

   Solve for T_2:

   $$T_2 = T_1 - \frac{Q\left(\frac{1}{k_1}\ln\frac{R_2}{R_1}\right)}{2\pi L}$$

   For our parameters and Q, we arrive at $T_2 = T_1 - 198^\circ\text{F} = 1802^\circ\text{F}$. Doing the same for the brick layer gives $T_3 = T_2 - 929^\circ\text{F} = 873^\circ\text{F}$. Finally, the analogue for the outer heat transfer coefficient is

   $$T_4 = T_3 - \frac{Q}{2\pi L h R_3}$$

   This gives $T_4 = T_3 - 803^\circ\text{F} = 70^\circ\text{F}$, which is the outside temperature, as it should be.

3. Polymer extrusion and thermal stress

   (a) The Biot number is given by:

   $$\text{Bi} = \frac{hR}{k} = \frac{130\frac{\text{W}}{\text{m}^2\cdot\text{K}} \cdot 0.01\text{m}}{\frac{\text{W}}{\text{m}\cdot\text{K}}} = 2.03$$

(b) For the Fourier number, we first convert z to time:

$$u_z = \frac{z}{t}, \text{ so } t = \frac{z}{u_z}$$

Fourier number definition:

$$\text{Fo} = \frac{\alpha t}{R^2} = \frac{kt}{\rho c_p R^2}$$

The length scale here is the radius; this was shown on the graphs on pages 715–716 of W^{3}C.

Distance z	time t	Fourier number
0.33 m	3.3 s	~ 0.01
1.0 m	10 s	~ 0.03
3.3 m	33 s	~ 0.1

(c) Since the Biot number is about 2, $m = \text{Bi}^{-1} = 0.5$, so we use the $m = 0.5$ curve in the graphs with $n = 0$ (center) and $n = 1$ (surface), which give the dimensionless temperatures and real temperatures as follows:

Distance	Fourier number	Center $\frac{T-T_f}{T_i-T_f}$	Center T	Surface $\frac{T-T_f}{T_i-T_f}$	Surface T
0.33 m	0.01	1.0	160°C	0.8	$\sim$ 135°C
1.0 m	0.03	1.0	160°C	0.7	$\sim$ 135°C
3.3 m	0.10	0.95	$\sim$ 155°C	0.5	100°C

The third of these, at $z = 3.3$m, obviously has the largest temperature difference.

(d) This is a Biot number issue, since lower Biot numbers correspond to more uniformity in the solid. Indeed, for a Biot number below 0.1, we have a "Newtonian cooling" a.k.a. "lumped parameter" situation where temperature is approximately uniform.

To reduce the Biot number $\frac{hR}{k}$, we can:

- use a different material with higher thermal conductivity k, though that would require, well, using a different material, but we have orders for HDPE rods.
- reduce the radius R, though that would require, well, reducing the radius, but we have orders for 2 cm diameter rods.
- reduce the heat transfer coefficient by turning off or slowing down some or all of the cooling fans, though this would require longer cooling time for the extruded rods, and therefore a longer line in the factory or a slower production rate. Either way, the product will be more costly, but that's better than shipping bad or out-of-spec product.

It's probably obvious that I was looking for the last of these three answers, but the question wording was vague enough that any of them would do.

Note however that slowing down the line alone would *not* improve temperature uniformity, it would just, well, slow down the line.

4. Cooling of a little plastic widget

Note that the ten seconds of free fall corresponds to 500 meters of height, so that part of the problem was just a bit unrealistic. Oh well.

(a) The Biot number is simply

$$\text{Bi} = \frac{hL}{k} = \frac{40\frac{\text{W}}{\text{m}^2\cdot\text{K}} \cdot 0.005\text{m}}{2.0\frac{\text{W}}{\text{m}\cdot\text{K}}} = 0.1.$$

The Newtonian Cooling assumption therefore applies, with uniform temperature across the widget.

(b) The Fourier number is

$$\text{Fo} = \frac{\alpha t}{L^2} = \frac{kt}{\rho c_p L^2} = \frac{2.0\frac{\text{W}}{\text{m}\cdot\text{K}} \cdot 10\text{s}}{900\frac{\text{kg}}{\text{m}^3} \cdot 2500\frac{\text{J}}{\text{kg}\cdot\text{K}} \cdot (0.005\text{m})^2} = 0.356.$$

(c) Because the Biot number is so small, the Newtonian cooling equation should apply; this equation is very accurate even for complex geometries like this one.

$$\frac{T - T_{fl}}{T_i - T_{fl}} = \exp\left(-\frac{Aht}{V\rho c_p}t\right)$$

$$T = T_{fl} + (T_i - T_{fl})\exp\left(-\frac{Aht}{V\rho c_p}\right)$$

$$T = 20°\text{C} + (160°\text{C} - 20°\text{C})\exp\left(-\frac{2\times 10^{-4}\text{m}^2 \cdot 40\frac{\text{W}}{\text{m}^2\cdot\text{K}} \cdot 10\text{s}}{5\times 10^{-8}\text{m}^3 \cdot 900\frac{\text{kg}}{\text{m}^3} \cdot 2500\frac{\text{J}}{\text{kg}\cdot\text{K}}}\right)$$

$$T = 20°\text{C} + 140°\text{C}\exp(-0.711) = 88.8°\text{C}$$

This temperature is the uniform temperature of the whole widget, and thus applies in the "center" and everywhere else.

(d) The thermal conductivity does not enter into the Newtonian cooling equation in part 4c, so it would appear that the final temperature would be unchanged.

However, lowering the thermal conductivity increases the Biot number (since k is in the denominator), which in part 4a was shown to be just at the Newtonian cooling threshold. The final temperature in this case will thus be slightly higher than was predited in part 4c.

Then again, a better estimate of the Biot number than in part 4a would use the volume/surface area ratio of 0.25 mm for L instead of the maximum dimension. In this case, the Biot number would go from 0.005 to 0.01, still raising the center temperature, but only very slightly.