Philosophy 244: #6—Testing: Decidable & Undecidable Systems

Validity has been defined in each case as validity for every frame of an appropriate type. Frames "of an appropriate type" are characterized by features of their accessibility relations: reflexive, transitive, serial, or what have you. A frame with transitive (or what have you) accessibility relation is said to be a transitive frame, and a model built on such a frame is said to be a transitive model.

How are we to test in particular cases whether a given wff α is valid on all, say, reflexive frames? One can't really run through all reflexive models and see whether α is true in each of these models' worlds. There are going to be an infinite number of models, due to the infinite number of (non-isomorphic) reflexive frames.

Now, as a matter of fact, for all of the systems we've considered so far, it suffices to consider frames of size less than or equal to N(α), where N is a number that varies with the complexity of the wff α This'll be shown later in the book, but in a chapter we're probably going to skip. As a practical matter this does not help anyway, for the number N is liable to get very large very quickly.

A shorter method, as we found too with propositional logic, is the reductio method: we hunt for a falsifying model, that is, a model where α is false at at least one world *w*.

Semantic Diagrams

Start with T. How do we test the wff $\alpha = (\Box p \ \& \ \Box(p \supset q)) \supset \Box q$. Well, spose there were a K-model with a world w_1 such that $V(\alpha, w_1)=0$. A certain amount can be done with the methods we already have.

($\Box$	p	&	$\Box$	(p	$\supset$	q))	$\supset$	$\Box$	q
1		1	1				0	0	
(3)		(2)	(4)				(1)	(3)	

This is as far as PC methods take us. But the rules for $\Box$ now kick in:

($\Box$	p	&	$\Box$	(p	$\supset$	q))	$\supset$	$\Box$	q
1	1	1	1	1	1	1	0	0	1
(3)	(5)	(2)	(4)	(7)	(6)	(8)	(1)	(3)	(9)

Asterisks are placed on *top* of $\Box$s if the wff is to be true, underneath if it is to be false. So

*			*						
($\Box$	p	&	$\Box$	(p	$\supset$	q))	$\supset$	$\Box$	q
1	1	1	1	1	1	1	0	0	1
								*	

Draw arrows up and down from the asterisks to any additional worlds they call for. That $\Diamond q$ is false at w_1 means it can see a world w_2 at which q is false. Let's add this second world to our diagram:

$\downarrow$

q	p	$p \supset q$
0	1	<u>!!!</u>

Note, we used reflexivity at steps (5) and (7)—-not essentially though so we have really proved K-validity.

A contradiction ensues at the place marked !!!; the value of $p \supset q$ at w_2 has got to be 1 since $\Box(p \supset q)$ holds in w_1, and likewise the value of p; but then q should be true in w_2 not false. So the wff is T-valid.

This is the *method of semantic diagrams*; the rules (apart from PC rules) are

I *Rule for asterisks*

An asterisk is put above every $\Box$ with a 1 beneath it and every $\Diamond$ with a 0 beneath it; an asterisk is put below every $\Box$ with a 0 beneath it and every $\Diamond$ with a 1 beneath it.

II *Rule for new worlds.*

A. If in *w* there occurs a formula $\Box\alpha$ with an asterisk above the $\Box$, then in every world accessible from *w*, α must be assigned 1.

B. If in *w* there occurs a formula $\Diamond\alpha$ with an asterisk above the $\Diamond$, then in some world accessible from *w*, α must be assigned 1.

C. If in *w* there occurs a formula $\Box\alpha$ with an asterisk below the $\Box$, there must be a world accessible from *w* in which $?\alpha$ is assigned 0.

D. If in *w* there occurs a formula $\Diamond\alpha$ with an asterisk below the $\Diamond$, then in every world accessible from *w*, ? must be assigned 0.

Let's go through an example or two from the book to see if we can get the hang of it..

The one remaining complication is to the "indeterministic" case. A disjunction stipulated to be true can be true in any of three ways so how are we supposed to fill in the component values. The book calls disjunction in this case a †-operator, and gives a special rule for them, that is hard to state but kind of what you'd expect.

III *Rule for alternatives*

If the rectangle for *w* contains a †-operator, split it into as many world-rectangles as you need to deal with all the ways its needs can be satisfied. Apart from resolvng the indeterminacy in these different ways, the new rectangles are just like the old. Proceed from left to right until all your rectangles are †-free, Arrows are drawn only from †-free rectangles.

When you've applied these rules all you can, you've got a complete diagram-system for α. A rectangle is inconsistent if the same subnormal is assigned both 0 and 1; or it demands access to an inconsistent rectangle; or all its alternatives are inconsistent rectangles.

THE TEST: α is valid if the original α=0 rectangle is inconsistent by these criteria.

Diagrams are fun, kind of, but also theoretically illuminating. To show T is decidable it suffices to show that the test always yields a verdict one way or the other. The worry of course is that you'll plug away forever, never arriving at a verdict of validity, but forever hopeful one will be reached tomorrow. How do we know this never happens? The formulas as we proceed keep on getting shorter and shorter.More exactly let a formula's modal degree is defined as follows:

Of course the implementation is going to be different if we're testing for T-validity or S4-validity. The stronger a system is, the more rectangles you need and the more values have to be written in. (We saw an example of this with reflexiity.) The more rectangles and values you have, the harder it becomes for that initial rectangle to maintain consistency.

1. PC-wffs are of degree 0.
2. degree($\neg\alpha$) = degree(α).
3. degree($\alpha\vee\beta$) = the larger of degree(α). degree(β).
4. degree($\Box\alpha$) = degree($\Diamond\alpha$) = degree(α)+1.

To check we've got it, what is the modal degree of $\Box(\Diamond p \vee \Diamond\Box(p\supset\Box p))$? It's 1 more than the modal degree of $(\Diamond p \vee \Diamond\Box(p\supset\Box p))$. Which is what? The larger of the modal degrees of $\Diamond p$ and $\Diamond\Box(p\supset\Box p)$. Which is...help me out here. The construction always terminates because if one rectangle points to another, its formulaes are of higher modal degree than the other's formulas. (Something like that.) All degrees are finite, so to get an infinite sequence of rectangles you'd need an infinite decreasing sequence of natural numbers.

Question 1: Is that right? Does it show that the process must terminate? Couldn't there still be infinite *branching*? Question 2: I said "all degrees are finite." That is true but is it necessary to the argument? Is an infinite decreasing sequence of arbitrarily large (finite or infinite) numbers possible?

Remember Milo's observation from the other other day.

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