Harvard-MIT Division of Health Sciences and Technology
HST.584J: Magnetic Resonance Analytic, Biochemical, and Imaging Techniques, Spring 2006
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HST.584 / 22.561 Problem Set #3 Solutions

Marking Scheme: Question 1 – 3 points, Question 2 – 3 points, Question 3 – 4 points

1) The easiest approach is to start from equilibrium and then figure out the magnetization components at the different time points. Tracking through the pulse sequence, we find the following components:

Event	$M_{longitudinal}$	$M_{transverse}$
Start at equilibrium	M_0	0
Apply π pulse	$-M_0$	0
TI Interval	$M_0 + (M_z(0) - M_0)e^{-TI/T1}$ $= M_0(1 - 2e^{-TI/T1})$	0
Apply π/2 pulse	0	$M_0(1 - 2e^{-TI/T1})$
TR – TI interval	$M_0(1 - e^{-(TR-TI)/T1})$	$M_0(1-2e^{-TI/T1})e^{-(TR-TI)/T2}$ $= \sim 0$
Apply π pulse	$-M_0(1 - e^{-(TR-TI)/T1})$	0
TI Interval	$M_0 + (-M_0(1 - e^{-(TR-TI)/T1}) - M_0)e^{-TI/T1}$ $= M_0(1 - 2e^{-TI/T1} + e^{-TR/T1})$	0
Apply π/2 pulse	Etc.	Etc.

So, in terms of the labels given in the question and assuming we have reached a steady-state:

Time	$M_{longitudinal}$	$M_{transverse}$
a	$M_0(1 - e^{-(TR-TI)/T1})$	0
a+	$-M_0(1 - e^{-(TR-TI)/T1})$	0
b	$M_0(1 - 2e^{-TI/T1} + e^{-TR/T1})$	0
b+	0	$M_0(1 - 2e^{-TI/T1} + e^{-TR/T1})$
c	$M_0(1 - e^{-(TR-TI)/T1})$	0

The signal is maximized immediately after the π/2 pulse (i.e. there are no echoes; our transverse magnetization simply decays away). The steady-state signal amplitude is thus $M_0(1 - 2e^{-TI/T1} + e^{-TR/T1})$

2-a) Signal amplitude is determined by our transverse component.

After θ_1: $M_z = M_0\cos 60 = M_0/2$, $M_{xy} = M_0\sin 60 = M_0\sqrt{3}/2$

Prior to θ_2: $M_z(TR^-) = M_z(0)e^{-1/5} + M_0(1 - e^{-1/5}) = 0.591M_0$, $M_{xy} = 0$

After θ_2: $M_z = M_z(TR^-)\cos 60 = 0.295\, M_0$, $M_{xy} = M_z(TR^-)\sin 60 = 0.512\, M_0$

2-b) For the 2nd FID to have zero amplitude, we require $M_z(TR^-) = 0$.

$M_z(TR^-) = M_0 \cos\theta_1\, e^{-TR/T1} + M_0(1 - e^{-TR/T1}) = 0$

$\theta_1 = 102.8°$

2-c) We now require $M_{xy}(0^+) = M_{xy}(TR^+)$. For q1 = 25°, $M_{xy}(0^+) = 0.423\ M_0$.

Therefore, $M_z(TR^-) = M_0 \cos 25\ e^{-1/5} + M_0 (1 - e^{-1/5}) = 0.923\ M_0$.

$M_{xy}(TR^+) = M_z(TR^-) \sin\theta_2 = M_{xy}(0^+)$.

$\theta_2 = 27.3°$

2-d) Ignoring T_1 effects, we find that $M_{xy}(0^+) = \sin\theta_1$, $Mxy(TR+) = \cos\theta_1 \sin\theta_2$. We wish to equate these and also maximize them. Clearly, the simplest solution for this is when $\theta_1 = 45°$, $\theta_2 = 90°$.

3) Rotation matrices can be used to track pulses and also phase evolutions. The rotation matrices are defined as:

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}, R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}, R_z(\theta) = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

All that remains is to write our pulse sequence in terms of these operations (and make sure we have the correct order, as the matrices DO NOT commute with each other). Accumulation of phase ϕ can be written as a rotation about the z-axis of angle ϕ.

a)

$$M_{net} = R_z(\phi) R_x(90) R_z(\phi) R_x(90) M_0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} M_0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= M_0 \begin{bmatrix} \cos\phi \sin\phi \\ -\sin^2\phi \\ -\cos\phi \end{bmatrix}$$

b) First, let's determine the effects of the $90°_x - \tau - 180°_x$ sequence:

$$M_{net} = R_z(\phi) R_x(180) R_z(\phi) R_x(180) M_0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} M_0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= M_0 \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$$

We see that we have all our transverse magnetization rephased after 2τ – this is just a standard spin-echo sequence. In part a, we found the two transverse components in terms of the angle ϕ; now we can just integrate those components across the uniform distribution stated in the question.

$$M_x = \frac{1}{2\pi}\int_0^{2\pi} M_0 \sin\phi\cos\phi d\phi = 0$$

$$M_y = \frac{1}{2\pi}\int_0^{2\pi} -M_0 \sin^2\phi d\phi = -\frac{M_0}{2}$$

c) Note: There is a typo in the question. The general sequence should have read θ_x – τ – $2\theta°_x$, not θ_x – τ – $\theta°_x$. I marked either answer correct however.

First, the correct sequence:

$$M_{net} = R_z(\phi)R_x(2\theta)R_z(\phi)R_x(\theta)M_0\begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$= \begin{bmatrix}\cos\phi & \sin\phi & 0\\ -\sin\phi & \cos\phi & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & \cos 2\theta & \sin 2\theta\\ 0 & -\sin 2\theta & \cos 2\theta\end{bmatrix}\begin{bmatrix}\cos\phi & \sin\phi & 0\\ -\sin\phi & \cos\phi & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\theta & \sin\theta\\ 0 & -\sin\theta & \cos\theta\end{bmatrix}M_0\begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$= M_0\begin{bmatrix}(\sin 2\phi + 2\sin\phi)\sin\theta\cos^2\theta\\ -\sin\theta + \sin\theta\cos^2\theta(2\cos\phi + 2\cos^2\phi)\\ \cos^3\theta - \sin^2\theta\cos\theta(1+2\cos\phi)\end{bmatrix}$$

Performing the same integration:

$$M_{xnet} = \frac{1}{2\pi}M_0\int_0^{2\pi}(\sin 2\phi + 2\sin\phi)\sin\theta\cos^2\theta d\phi = 0$$

$$M_{ynet} = \frac{1}{2\pi}M_0\int_0^{2\pi} -\sin\theta + \sin\theta\cos^2\theta(2\cos\phi + 2\cos^2\phi)d\phi = -M_0\sin^3\theta$$

Now, with the typo sequence given in the question:

$$M_{net} = R_z(\phi)R_x(\theta)R_z(\phi)R_x(\theta)M_0\begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$= \begin{bmatrix}\cos\phi & \sin\phi & 0\\ -\sin\phi & \cos\phi & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\theta & \sin\theta\\ 0 & -\sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}\cos\phi & \sin\phi & 0\\ -\sin\phi & \cos\phi & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\theta & \sin\theta\\ 0 & -\sin\theta & \cos\theta\end{bmatrix}M_0\begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$= M_0\begin{bmatrix}\frac{1}{2}\sin 2\phi(\sin\theta + \frac{1}{2}\sin 2\theta) + \frac{1}{2}\sin\phi\sin 2\theta\\ -\sin^2\phi\sin\theta + \frac{1}{2}\sin 2\theta(\cos\phi + \cos^2\phi)\\ 1 - \sin^2\theta(1+\cos\phi)\end{bmatrix}$$

Performing the same integration:

$$M_{xnet} = \frac{1}{2\pi}M_0\int_0^{2\pi}[\tfrac{1}{2}\sin 2\phi(\sin\theta + \tfrac{1}{2}\sin 2\theta) + \tfrac{1}{2}\sin\phi\sin 2\theta]d\phi = 0$$

$$M_{ynet} = \frac{1}{2\pi}M_0\int_0^{2\pi}[-\sin^2\phi\sin\theta + \tfrac{1}{2}\sin 2\theta(\cos\phi + \cos^2\phi)]d\phi = \frac{M_0}{4}(\sin 2\theta - 2\sin\theta)$$