6.849	Lecture 7	Sept. 29, 2010

Fold & one cut:

1. fold flat
2. make one complete straight cut
3. unfold

— what shapes/patterns of cuts are possible?

History:

- Kan Chu Sen [1721] — Japanese puzzle book "Wakoku Chiyekurabe"
- Betsy Ross [1873 story] — ☆ in American flag
- Harry Houdini [1922] — ☆ in Paper Magic [ghostw.]
- Gerald Loe [1955] — Paper Capers ~ simple folds
- Martin Gardner [1960] — Scientific American ~ OPEN

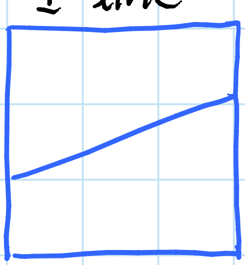
Universality: any set of line segments can be cut

— two methods:

1. straight skeleton [Demaine, Demaine, Lubiw 1998]
   — works almost always; practical
2. disk packing [Bern, Demaine, Eppstein, Hayes 1998]
   — always works; pseudopolynomial; impractical

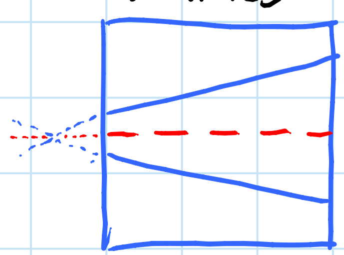
Warm-ups:

1 line



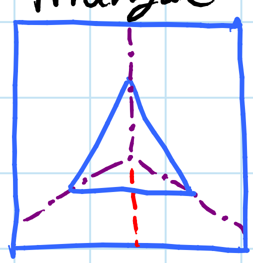
no folds

2 lines



bisector

triangle

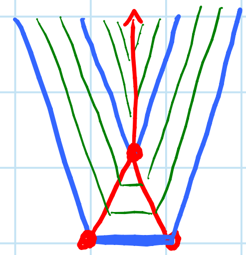


angle bisect + perp.

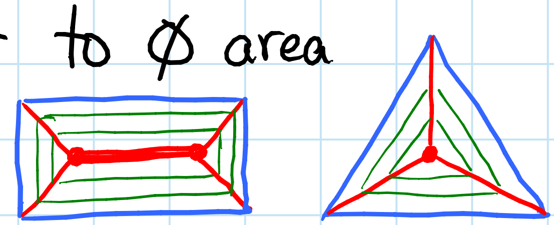
Straight skeleton: [Aichholzer et al. 1995 & 1996]

= trajectory of the vertices of the desired cut pattern as we simultaneously <u>shrink</u> each region, keeping edges parallel to the originals & at uniform perpendicular distance

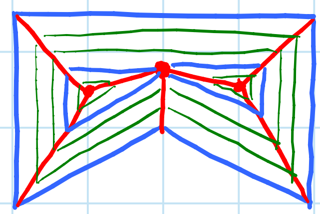
Events during shrinking:



① edge shrinks to ∅ length
⇒ drop it

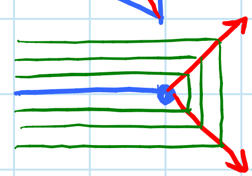


② entire region collapses to ∅ area
⇒ add it all
& stop shrinking it

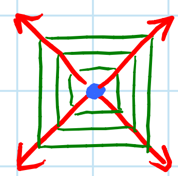


③ face "splits"
⇒ recurse in pieces

Degree-1 vertex like end of a rectangle



Degree-∅ vertex like a square



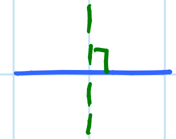
Facts:

- $O(n)$ skeleton vertices, edges, regions
- one-to-one correspondence between <u>cut edges</u> and regions of the <u>straight skeleton</u>
- every <u>skeleton edge</u> is a subsegment of the (angular) bisector of the <u>cut edges</u> corresponding to the two incident <u>skeleton regions</u> ⇒ align

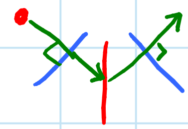
Generic skeleton vertex has degree 3 ⇒ not flat foldable
⇒ need to add some creases...

Perpendiculars: [Demaine, Demaine, Lubiw 1998]

add creases that meet desired cuts at right angles ⇒ preserve alignment



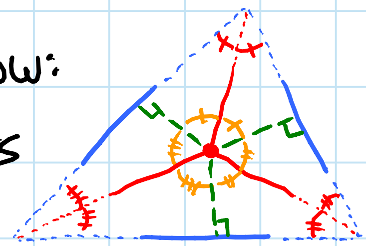
- from each skeleton vertex, try to enter each incident skeleton region with ray perpendicular to corresponding cut edge



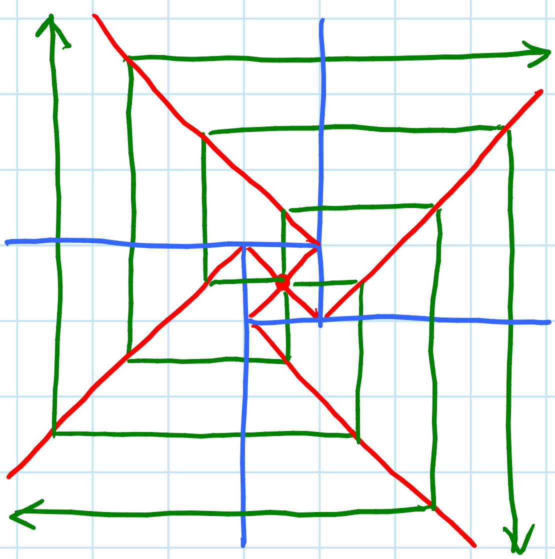
- if ray hits another skeleton edge, reflect (⇒ remains perpendicular to corresponding cut edge)

Typical behavior at skeleton vertex now:
- skeleton edges bisect perpendiculars

⇒ Kawasaki condition holds

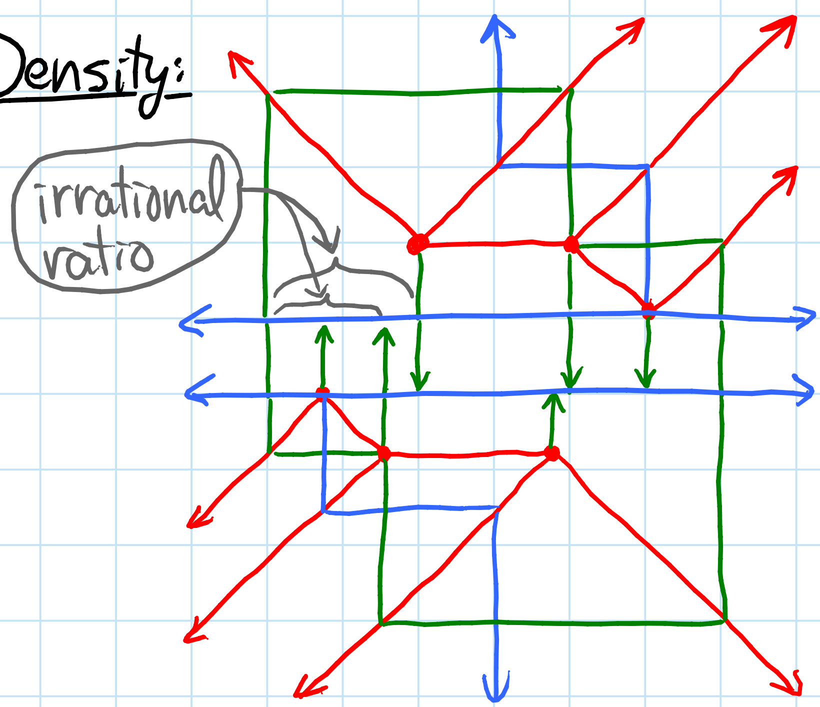


Spiraling:



⇒ infinite creases, but finite in finite paper

Density:



⇒ creases are dense

$\boxed{\text{CONJECTURE}}$: rare (prob. ∅)

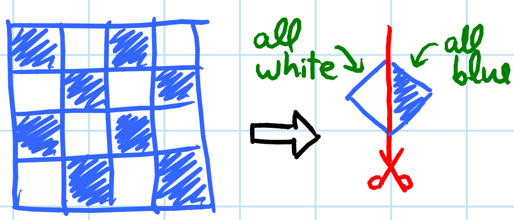
Mountain-valley assignment: (initial)

- skeleton edge mountain if bisects convex angle
  valley if bisects reflex angle
- cut edge valley
- perpendiculars alternate mountain/valley;
  start to be determined later

Side assignment: specify which cut regions are above or below the cut line

- skeleton edges as above in above regions;
  reversed in below regions
- cut edge valley between two above regions
  mountain between two below regions
  uncreased between one above & one below

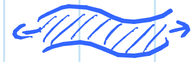
- e.g. 2-regular (nested/disjoint polygons)
  ⇒ natural 2-coloring
  ⇒ all cuts uncreased ("scissor cuts")
- e.g. 4-regular checkerboard

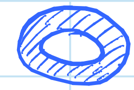


PROJECT: implement crease pattern algorithm (ideally with degeneracy tool, M/V assignment, folded state...)

Send me your cool fold & cut examples!

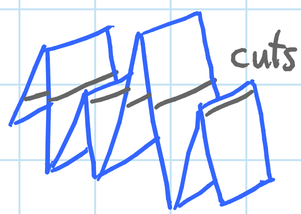
Corridor = region delimited by perpendiculars (like rivers)

- constant width, measured perpendicularly
- either **linear** = eventually reach infinity 

  or **circular** = closed loop 

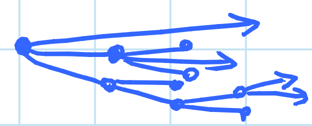
  ↳ harder to fold (theoretically & practically)
- $\boxed{\text{CONJECTURE}}$: max. degree 2 ⇒ linear corridors only with probability 1

Linear-corridor case: (proof sketch)



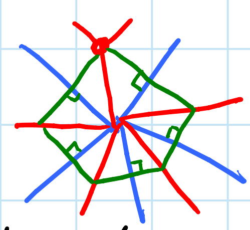
- each corridor folds as an accordion
  - alternates mountain/valley
  - aligns cut edges
- corridors form a tree structure ≈ projection
  - edge = corridor, length = width
  - vertex = connected component of perpendiculars
- fold tree flat by depth-first search

  ⇒ origami folds flat (argue noncrossing)

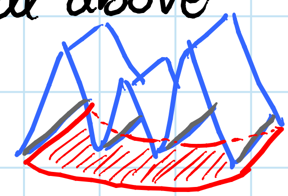


Circular-corridor case:

- **trouble**: accordion has to wrap around at some edge — reversed: intersect?
- $\boxed{\text{CONJECTURE}}$: with probability 1, circular corridors are **normal** →



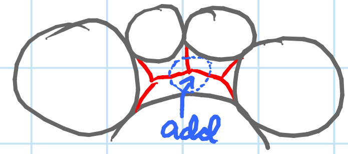
- if normal & side assignment is "all above" then can reverse a cut edge:



Disk-packing method: [Bern, Demaine, Eppstein, Hayes 1998–2006] & O'Rourke

① thicken desired cuts by 2ε
by parallel offset by $\pm\varepsilon$ (ε suff. small)
(just like straight skeleton)

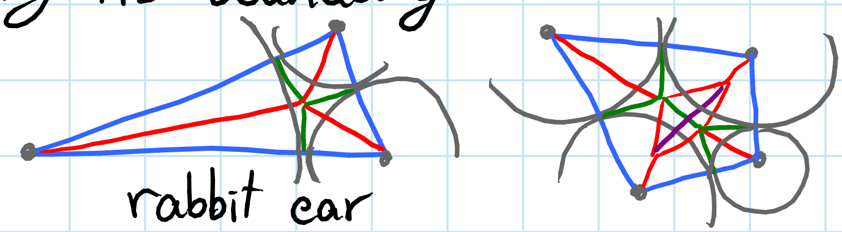
② find a (nonoverlapping) disk packing such that
ⓐ every vertex of offset cuts & paper boundary is the center of a disk
– put small disk at each vertex
ⓑ every edge of ... is a union of radii
– pack small disks along each edge
ⓒ every gap between disks has 3 or 4 sides
– repeatedly subdivide gaps:



[Eppstein 1997]

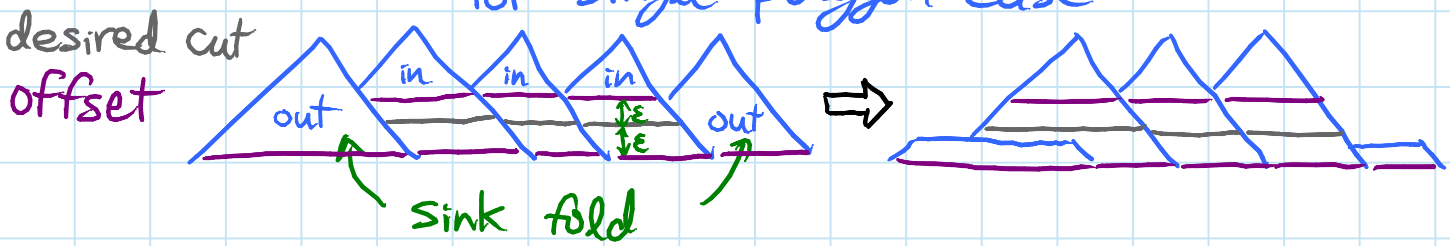
③ dual ⇒ decomposition into triangles & quads.

④ fold each triangle/quad. into molecule aligning its boundary



⑤ glue molecules together ⇒ align all edges
– argue no crossings ~ hard part

⑥ sink-fold exterior molecules to height $<\varepsilon$
for single polygon case



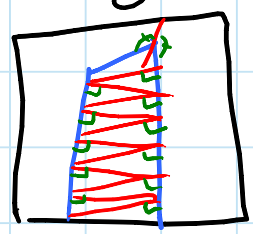
Disk-packing method: (cont'd)

- can generalize to arbitrary cut graphs
  (but not arbitrary side assignments)
  - joining & sinking gets messier
- can bound # creases (# disks) in terms of n & integral of "local feature size"
  (distance from x to another boundary point, dx)

$\boxed{\text{OPEN}}$: strongly polynomial bound possible for any solution to fold & cut?
(conjecture not...)

Simple fold & cut: [Demaine, Demaine, Hawksley, Ito, Loh, Manber, Stephens 2010]

- all layers: (strongly) polynomial-time algorithm for polygons with margin
  - but # folds can be arb. large:



  - idea: guess a line of symmetry
    fold down to convex hull
    make "best possible" safe fold
    (reduce # vertices if possible)
    ⇒ graph gets smaller (smaller ⊆ larger)
    - here use polygonness
    ⇒ convex hull (paper) gets smaller
- all layers: convex polygon ⟺ line of symmetry
- some layers: xy-monotone orthogonal polygons

Flattening polyhedra: given polyhedral surface as piece of paper, can it fold flat at all? (without tearing)
[Demaine, Demaine, Lubiw 2000]

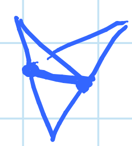
Connection to fold & cut:

	2D fold & cut	3D fold & cut
— paper:	2D region	3D solid
— cuts:	1D segments	2D polygons
— fold:	through 3D	through 4D
— flat:	down to 2D	down to 3D
— so that:	segments on line	polygons on plane

⇒ flattening is boundary of 3D fold & cut

OPEN: d-D fold & cut for $d \geq 3$? e.g. convex polyh.?
OPEN: align all k-D faces, $0 \leq k \leq d$, for $d \geq 2$?
OPEN: flattening based on 3D straight skeleton? [Demaine, Demaine, Lubiw 2000]
— possible for "thin convex prismatoids"

Flat folded state exists for orientable manifolds [Bern & Hayes 2008]
— based on disk packing fold & cut [see Ch. 18]



OPEN: arbitrary polyhedral complexes
OPEN: continuous motion?
OPEN: connected configuration space of a polyhedral piece of paper?
~ no canonical state — not possible rigidly