6.642 Continuum Electromechanics
Fall 2008

Massachusetts Institute of Technology
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6.642 Continuum Electromechanics

Problem Set #7 | Issued: 11/12/08
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Problem 1

Prob. 8.12.2 (Melcher, Continuum Electromechanics)

Problem 2

Prob. 8.13.2 (Melcher, Continuum Electromechanics)

Prob. 8.12.1 (continued)

(a) Determine the equilibrium difference in pressure between the regions a and b and the fluid o.

(b) Show that deflections of the interfaces can be divided into kink modes $[\xi^a(y,z,t) = \xi^b(y,z,t)]$, and sausage modes $[\xi^a(y,z,t) = -\xi^b(y,z,t)]$.

(c) Show that the dispersion equation for the kink modes is*, with $k \equiv \sqrt{k_y^2 + k_z^2}$,

$$\frac{\rho\omega^2}{k}\tanh\left(\frac{kd}{2}\right) = \mu_o H_o^2 \frac{k_z^2}{k}\coth(ka)$$

while the dispersion equation for the sausage modes is

$$\frac{\rho\omega^2}{k}\coth\left(\frac{kd}{2}\right) = \mu_o H_o^2 \frac{k_z^2}{k}\coth(ka)$$

(d) Is the equilibrium, as modeled, stable? The same conclusion should follow from both the analytical results and intuitive arguments.

Prob. 8.12.2 At equilibrium, a perfectly conducting fluid (plasma) occupies the annular region $R < r < a$ (Fig. P8.12.2.) It is bounded on the outside by a rigid wall at $r = a$ and on the inside by free space. Coaxial with the annulus is a "perfectly" conducting rod of radius b. Current passing in the z direction on this inner rod is returned on the plasma interface in the -z direction. Hence, so long as the interface is in equilibrium, the magnetic field in the free-space annulus $b < r < R$ is

$$\vec{H} = H_o \frac{R}{r}\vec{i}_\theta$$

(a) Define the pressure in the region occupied by the magnetic field as zero. What is the equilibrium pressure Π in the plasma?

(b) Find the dispersion equation for small-amplitude perturbations of the fluid interface. (Write the equation in terms of the functions $F(\alpha,\beta)$ and $G(\alpha,\beta)$.)

(c) Show that the equilibrium is stable.

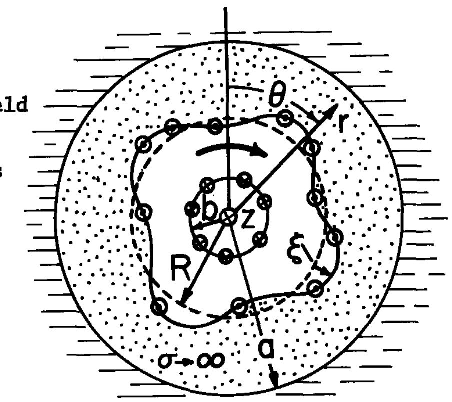

Fig. P8.12.2

Prob. 8.12.3 A "perfectly" conducting incompressible inviscid liquid layer rests on a rigid support at $x = -b$ and has a free surface at $x = \xi$. At a distance a above the equilibrium interface $\xi=0$ is a thin conducting sheet having surface conductivity σ_s. This sheet is backed by "infinitely" permeable material. The sheet and backing move in the y direction with the imposed velocity U. With the liquid in static equilibrium, there is a surface current $K_z = -H_o$ in the conducting sheet that is returned on the interface of the liquid. Thus, there is an equilibrium magnetic field intensity $\vec{H} = H_o\vec{i}_y$ in the gap between liquid and sheet. Include in the model gravity acting in the -x direction and surface tension. Determine the dispersion equation for temporal or spatial modes.

Prob. 8.12.4 In the pinch configuration of Fig. 8.12.1, the wall at r=a consists of a thin conducting shell of surface conductivity σ_s (as described in Sec. 6.3) surrounded by free space.

(a) Find the dispersion equation for the plasma column coupled to this lossy wall.

(b) Suppose that the frequencies of modes have been found under the assumption that the wall is perfectly conducting. Under what condition would these frequencies be valid for the wall of finite conductivity?

(c) Now suppose that the wall is very lossy. Show that the dispersion equation reduces to a quadratic expression in $(j\omega)$ and show that the wall tends to induce damping.

For Section 8.13:

Prob. 8.13.1 A cylindrical column of liquid, perhaps water, of equilibrium radius R, moves with uniform equilibrium velocity U in the z direction, as shown in Fig. P8.13.1. A coaxial cylindrical electrode is used to impose a radially symmetric electric field intensity

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* $\coth kd - \dfrac{1}{\sinh kd} \equiv \tanh\left(\dfrac{kd}{2}\right)$; $\coth kd + \dfrac{1}{\sinh kd} \equiv \coth\left(\dfrac{kd}{2}\right)$

Melcher, James R. *Continuum Electromechanics*. Cambridge, MA: MIT Press, 1981. ISBN: 9780262131650.

Prob. 8.13.1 (continued)

$$\vec{E} = E_o \frac{R}{r} \vec{i}_r$$

in the region between the electrode and liquid.

Assume that the density of the liquid is large compared to that of the surrounding gas. Moreover, consider the liquid to have a relaxation time short compared to any other times of interest, and assume that the cylindrical electrode is well removed from the surface of the liquid.

(a) Determine the equilibrium pressure jump at the interface.

(b) Show that the dispersion equation is

$$(\omega - kU)^2 = \frac{\gamma}{\rho R^3}[-Rf_m(0,R)]\left\{m^2 - 1 + (kR)^2 + \frac{\varepsilon_o E^2 R}{\gamma}[1 - Rf(\infty,R)]\right\}$$

by using the transfer relations of Tables 2.16.2 and 7.9.1.

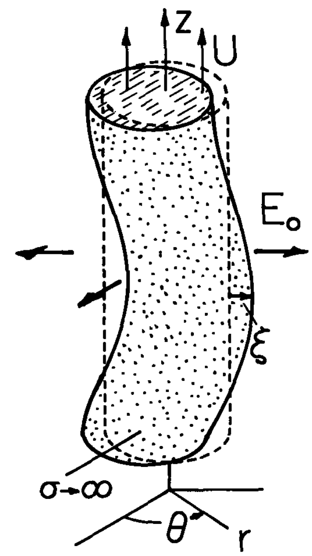

Fig. P8.13.1

Prob. 8.13.2 A spherical drop of insulating liquid is of radius R and permittivity ε. At its center is a metallic, spherical particle of radius $b < R$ supporting the charge q. Hence, in equilibrium, the drop is stressed by a radial electric field.

(a) What is the equilibrium $\vec{E}$ in the drop ($b < r < R$) and in the surrounding gas, where the mass density is considered negligible and $\varepsilon \simeq \varepsilon_o$?

(b) Determine the dispersion equation for perturbations from the equilibrium.

(c) What is the maximum q consistent with stability for $b \ll R$?

For Section 8.14:

Prob. 8.14.1 For a conducting drop, such as water in air, the model of Sec. 8.13, where the drop is pictured as perfectly conducting, is appropriate. Here, the drop is pictured as perfectly insulating with charge distributed uniformly over its volume. The goal is to find the limit on the net drop charge consistent with stability; i.e., the analogue of Rayleigh's limit. This model is of historical interest because it was used as a starting point in the formulation of the liquid drop model of the nucleus.[2] In fact, the term in that model from nuclear physics that accounts for fission is motivated by the effect of a uniform charge density. Assume that the drop is uniformly charged, has a net charge Q but has permittivity equal to that of free space. Find the maximum charge consistent with stability.

Prob. 8.14.2 Consider the same configuration as developed in this section with the following generalization. The fluids in the upper and lower regions have permittivities ε_a and ε_b respectively.

(a) Write the equilibrium and perturbation bulk and boundary conditions.

(b) Find the dispersion equation and discuss the implications of the terms.

For Section 8.15:

Prob. 8.15.1 This problem is similar to that treated in the section. However, the magnetic field is imposed and the motions are two-dimensional, so that it is possible to represent the magnetic force density as the gradient of a scalar. This makes the analysis much simpler. A column of liquid-metal carries the uniform current density J_o in the z direction but suffers deformations that are independent of z. A wire at the center of the column also carries a net current I along the z axis. The field associated with this current is presumed much greater than that due to J_o. Thus, self fields due to J_o are ignored. Assume that the wire provides a negligible mechanical constraint on the motion and that the mass density of the gas surrounding the column is much less than that of the column.

(a) Show that the magnetic force density is of the form $-\nabla\mathcal{E}$, where

$$\mathcal{E} = -\frac{\mu_o I}{2\pi} \ln\left(\frac{r}{R}\right)$$

2. I. Kaplan, Nuclear Physics, Addison-Wesley Publishing Company, Reading, Mass., 1955, p. 425.