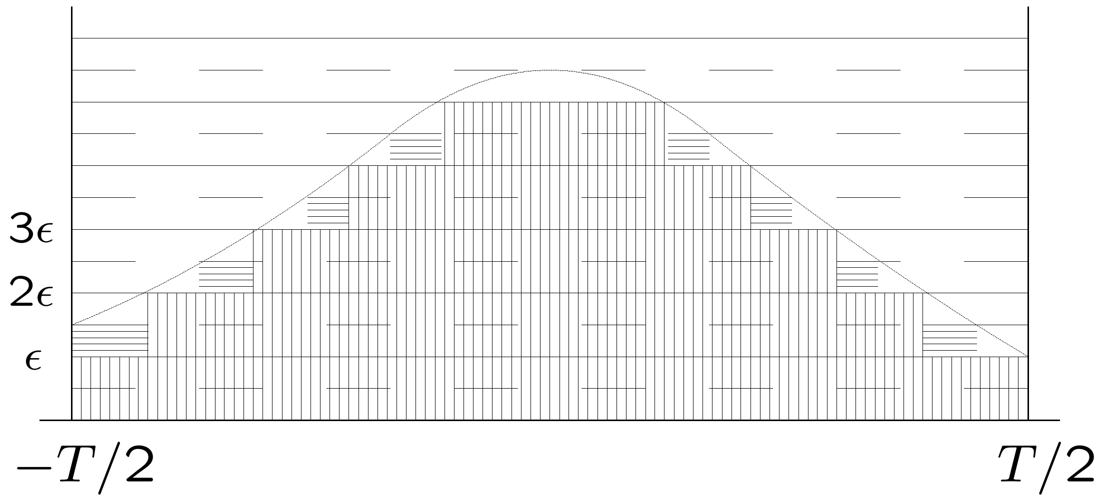
**A function $\{u(t) : \mathbb{R} \to \mathbb{R}\}$ is measurable if $\{t : u(t) < b\}$ is measurable for each $b \in \mathbb{R}$.**

**The Lebesgue integral exists if the function is measurable and if the limit in the figure exists.**




**Horizontal crosshatching is what is added when $\varepsilon \to \varepsilon/2$. For $u(t) \geq 0$, the integral must exist (with perhaps an infinite value).**

$\mathcal{L}_2$ **functions** $[-T/2, T/2] \to \mathbb{C}$

$\mathcal{L}_1$ **functions** $[-T/2, T/2] \to \mathbb{C}$

**Measurable functions** $[-T/2, T/2] \to \mathbb{C}$

$t^{-2/3}$ **for** $0 < t \leq T$ **is** $\mathcal{L}_1$ **but not** $\mathcal{L}_2$

**But for functions from** $\mathbb{R} \to \mathbb{C}$**,** $t^{-2/3}$ **for** $t > 1$ **is** $\mathcal{L}_2$ **but not** $\mathcal{L}_1$**. No general rule for** $\mathbb{R} \to \mathbb{C}$**.**

**Theorem: Let $\{u(t) : [-T/2, T/2] \to \mathbb{C}\}$ be an $\mathcal{L}_2$ function. Then for each $k \in \mathbb{Z}$, the Lebesgue integral**

$$\hat{u}_k = \frac{1}{T} \int_{-T/2}^{T/2} u(t)\, e^{-2\pi ikt/T}\, dt$$

**exists and satisfies $|\hat{u}_k| \le \frac{1}{T} \int |u(t)|\, dt < \infty$. Furthermore,**

$$\lim_{k_0 \to \infty} \int_{-T/2}^{T/2} \left| u(t) - \sum_{k=-k_0}^{k_0} \hat{u}_k\, e^{2\pi ikt/T} \right|^2 dt = 0,$$

**where the limit is monotonic in $k_0$.**

$$u(t) = \text{l.i.m.} \sum_k \hat{u}_k\, e^{2\pi ikt/T} dt$$

**The functions $e^{2\pi ikt/T}$ are orthogonal.**

**If an arbitrary $\mathcal{L}_2$ function $u(t)$ is segmented into $T$ spaced segments, the following expansion follows:**

$$u(t) = \text{l.i.m.} \sum_{m,k} \hat{u}_{k,m}\, e^{2\pi i k t/T}\, \text{rect}(\frac{t}{T} - m)$$

**This called the $T$-spaced truncated sinusoid expansion. It expands a function into time/frequency slots, $T$ in time, $1/T$ in frequency.**

## $\mathcal{L}_2$ functions and Fourier transforms

$$\hat{v}_A(f) = \int_{-A}^{A} u(t)e^{-2\pi ift}\ dt.$$

**Plancherel 1: There is an $\mathcal{L}_2$ function $\hat{u}(f)$ (the Fourier transform of $u(t)$), which satisfies the energy equation and**

$$\lim_{A\to\infty} \int_{-\infty}^{\infty} |\hat{u}(f) - \hat{v}_A(f)|^2\ dt = 0.$$

**Although $\{\hat{v}_A(f)\}$ is continuous for all $A \in \mathbb{R}$, $\hat{u}(f)$ is not necessarily continuous. We denote this function $\hat{u}(f)$ as**

$$\hat{u}(f) = \text{l.i.m.} \int_{-\infty}^{\infty} u(t)e^{2\pi ift}\, dt.$$

**For the inverse transform, define**

$$u_B(t) = \int_{-B}^{B} \hat{u}(f) e^{2\pi i f t}\ df \tag{1}$$

**This exists for all $t \in \mathbb{R}$ and is continuous.**

**Plancherel 2: For any $\mathcal{L}_2$ function $u(t)$, let $\hat{u}(f)$ be the FT of Plancherel 1. Then**

$$\lim_{B \to \infty} \int_{-\infty}^{\infty} |u(t) - w_B(t)|^2\ dt = 0. \tag{2}$$

$$u(t) = \text{l.i.m.} \int_{-\infty}^{\infty} \hat{u}(f) e^{2\pi i f t}\, df$$

**<u>All</u> $\mathcal{L}_2$ functions have Fourier transforms in this sense.**

**The DTFT (Discrete-time Fourier transform) is the t$\leftrightarrow$f dual of the Fourier series.**

**Theorem (DTFT) Assume** $\{\hat{u}(f) : [-W, W] \to \mathbb{C}\}$ **is** $\mathcal{L}_2$ **(and thus also** $\mathcal{L}_1$**). Then**

$$u_k = \frac{1}{2W} \int_{-W}^{W} \hat{u}(f) e^{2\pi i k f/(2W)} \, df$$

**is a finite complex number for each** $k \in \mathbb{Z}$**. Also**

$$\lim_{k_0 \to \infty} \int_{-W}^{W} \left| \hat{u}(f) - \sum_{k=-k_0}^{k_0} u_k \, e^{-2\pi i k f/(2W)} \right|^2 df = 0,$$

$$\hat{u}(f) = \text{l.i.m.} \sum_k u_k e^{-2\pi i f t/(2W)} \text{rect}\left(\frac{f}{2W}\right)$$

**Sampling Theorem: Let $\{\hat{u}(f) : [-W\,W] \to \mathbb{C}\}$ be $\mathcal{L}_2$ (and thus also $\mathcal{L}_1$). For inverse transform, $u(t)$, let $T = 1/(2W)$. Then $u(t)$ is continuous, $\mathcal{L}_2$, and bounded by $u(t) \leq \int_{-W}^{W} |\hat{u}(f)|\,df$. Also, for all $t \in \mathbb{R}$,**

$$u(t) = \sum_{k=-\infty}^{\infty} u(kT)\ \mathsf{sinc}\left(\frac{t-kT}{T}\right).$$

**The sampling theorem holds only for the inverse transform of $\hat{u}(f)$, not for the $\mathcal{L}_2$ equivalent functions whose transform is $\hat{u}(f)$.**

$$\hat{u}(f) = \sum_k u_k e^{-2\pi i k \frac{f}{2W}} \, \text{rect}\left(\frac{f}{2W}\right)$$

$$u_k = \frac{1}{2W} \int_{-W}^{W} \hat{u}(f) e^{2\pi i k \frac{f}{2W}} \; df$$

$$\begin{tikzcd}
\boxed{\textbf{Fourier series}} \arrow[r, shift left] & \boxed{\textbf{DTFT}} \arrow[l, shift left, "\textbf{T/F dual}"] \arrow[d, shift left, "\textbf{Fourier transform}"] \\
& \boxed{\textbf{Sampling}} \arrow[u, shift left]
\end{tikzcd}$$

$$u(t) = \sum_{k=-\infty}^{\infty} \hat{u}_k \, e^{2\pi i k t/T} \, \text{rect}\left(\frac{t}{T}\right)$$

$$\hat{u}_k = \frac{1}{T} \int_{-T/2}^{T/2} u(t) e^{-2\pi i k t/T} \, dt$$

$$u(t) = \sum_{k=-\infty}^{\infty} 2W u_k \, \text{sinc}(2Wt - k)$$

$$u_k = \frac{1}{2W} u\left(\frac{k}{2W}\right)$$

**Breaking $\hat{u}(f)$ into frequency segments, we get the T-spaced sinc-weighted sinusoid expansion,**

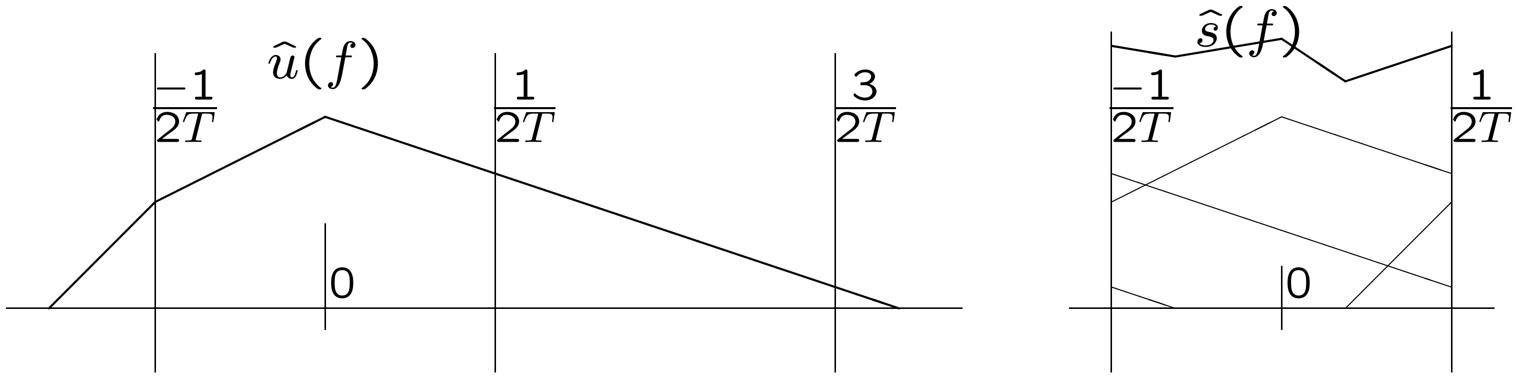
$$u(t) = \text{l.i.m.} \sum_{m,k} v_m(kT)\, \text{sinc}\left(\frac{t}{T} - k\right)\, e^{2\pi i m t/T}.$$

**Both this and the T-spaced truncated sinusoid expansion**

$$u(t) = \text{l.i.m.} \sum_{m,k} \hat{u}_{k,m}\, e^{2\pi i k t/T} \text{rect}\left(\frac{t}{T} - m\right)$$

**break the function into increments of time duration $T$ and frequency duration $1/T$.**

**For time interval $[-T_0/2, T_0/2]$ and frequency interval $[-W_0, W_0$, we get $2T_0W_0$ complex degrees of freedom.**




**Theorem: Let $\hat{u}(f)$ be $\mathcal{L}_2$, and satisfy**

$$\lim_{|f|\to\infty} \hat{u}(f)|f|^{1+\varepsilon} = 0 \qquad \textbf{for}\, \varepsilon > 0.$$

**Then $\hat{u}(f)$ is $\mathcal{L}_1$, and the inverse transform $u(t)$ is continuous and bounded. For $T > 0$, the sampling approx. $s(t) = \sum_k u(kT)\,\mathrm{sinc}(\frac{t}{T} + k)$ is bounded and continuous. $\hat{s}(f)$ satisfies**

$$\hat{s}(f) = \text{l.i.m.} \sum_m \hat{u}(f + \frac{m}{T})\,\mathrm{rect}[fT].$$

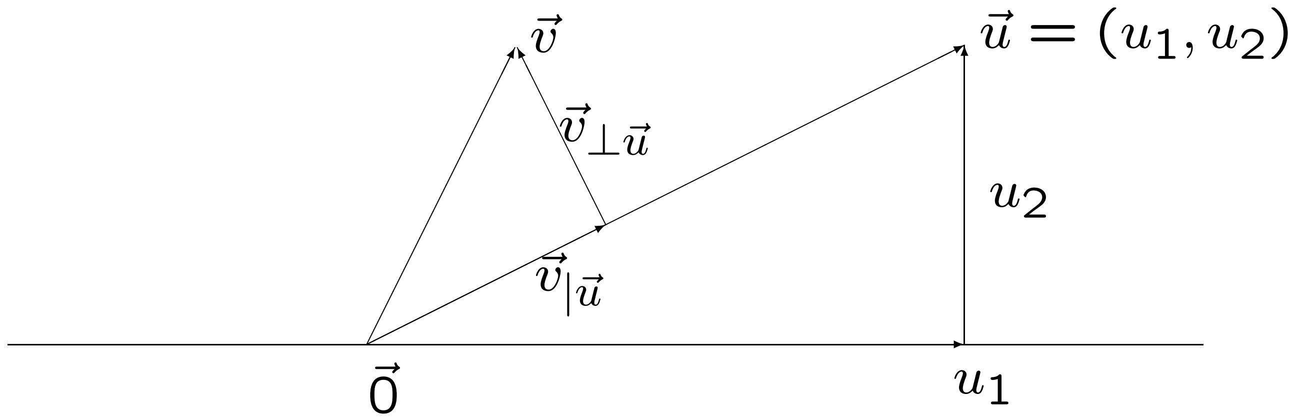
**$\mathcal{L}_2$ is an inner product space with the inner product**

$$\langle \vec{u}, \vec{v} \rangle = \int_{-\infty}^{\infty} u(t) v^*(t) dt,$$

**Because $\langle \vec{u}, \vec{u} \rangle \neq 0$ for $\vec{u} \neq 0$, we must define equality as $\mathcal{L}_2$ equivalence.**

**The vectors in this space are equivalence classes.**

**Alternatively, view a vector as a set of coefficients in an orthogonal expansion.**




**Theorem: (1D Projection) Let $\vec{v}$ and $\vec{u} \neq 0$ be arbitrary vectors in a real or complex inner product space. Then there is a unique scalar $\alpha$ for which $\langle \vec{v} - \alpha \vec{u}, \vec{u} \rangle = 0$. That $\alpha$ is given by $\alpha = \langle \vec{v}, \vec{u} \rangle / \|\vec{u}\|^2$.**

$$\vec{v}_{|\vec{u}} = \frac{\langle \vec{v}, \vec{u} \rangle}{\|\vec{u}\|^2} \vec{u} = \langle \vec{v}, \frac{\vec{u}}{\|\vec{u}\|} \rangle \frac{\vec{u}}{\|\vec{u}\|}$$

**Projection theorem: Assume that $\{\phi_1, \dots, \phi_n\}$ is an orthonormal basis for an $n$-dimensional subspace $\mathcal{S} \subset \mathcal{V}$. For each $\vec{v} \in \mathcal{V}$, there is a unique $\vec{v}_{|\mathcal{S}} \in \mathcal{S}$ such that $\langle \vec{v} - \vec{v}_{|S}, \vec{s} \rangle = 0$ for all $\vec{s} \in \mathcal{S}$. Furthermore,**

$$\vec{v}_{|\mathcal{S}} = \sum_j \langle \vec{v}, \phi_{\mathbf{j}} \rangle \, \phi_{\mathbf{j}}.$$

$$0 \le \|\vec{v}_{|\mathcal{S}}\|^2 \le \|\vec{v}\|^2 \quad \textbf{(Norm bounds)}$$

$$0 \le \sum_{j=1}^{n} |\langle \vec{v}, \boldsymbol{\phi}_j \rangle|^2 \le \|\vec{v}\|^2 \quad \textbf{(Bessel's inequality)}.$$

$$\|\vec{v} - \vec{v}_{|\mathcal{S}}\| \le \|\vec{v} - \vec{s}\| \quad \textbf{for any } \vec{s} \in \mathcal{S} \quad \textbf{(LS property)}.$$

For $\mathcal{L}_2$, the projection theorem can be extended to a countably infinite dimension.

Given any orthogonal set of functions $\theta_i$, we can generate orthonormal functions as $\phi_i = \boldsymbol{\theta}_i / \|\boldsymbol{\theta}_i\|$.

Theorem: Let $\{\phi_m, 1 \leq m < \infty\}$ be a set of orthonormal functions, and let $\vec{v}$ be any $\mathcal{L}_2$ vector. Then there is a unique $\mathcal{L}_2$ vector $\vec{u}$ such that $\vec{v} - \vec{u}$ is orthogonal to each $\phi_n$ and

$$\lim_{n \to \infty} \|\vec{u} - \sum_{m=1}^{n} \langle \vec{v}, \boldsymbol{\phi}_m \rangle \boldsymbol{\phi}_m\| = 0.$$

This "explains" convergence of orthonormal expansions.

**Since ideal Nyquist is all about samples of $g(t)$, we look at aliasing again. The baseband reconstruction $s(t)$ from $\{g(kT)\}$ is**

$$s(t) = \sum_k g(kT)\mathrm{sinc}(\frac{t}{T} - k).$$

**$g(t)$ is ideal Nyquist iff $s(t) = \mathrm{sinc}(t/T)$ i.e., iff**

$$\hat{s}(f) = T\,\mathrm{rect}(fT)$$

**From the aliasing theorem,**

$$\hat{s}(f) = \sum_m \hat{g}(f + \frac{m}{T})\,\mathrm{rect}(fT).$$

**Thus $g(t)$ is ideal Nyquist iff**

$$\sum_m \hat{g}(f + m/T)\,\mathrm{rect}(fT) = T\,\mathrm{rect}(fT)$$

$\vec{Z} = (Z_1, \ldots, Z_k)^{\mathsf{T}}$ **is zero-mean jointly Gauss iff**

- $\vec{Z} = \mathbf{A}\vec{N}$ **for normal** $\vec{N}$**.**

- $\vec{Z} = \mathbf{A}\vec{Y}$ **for zero-mean jointly-Gauss** $\vec{Y}$**.**

- **All linear combinations of** $\vec{Z}$ **are Gaussian.**

**Also linearly-independent iff**

- $f_{\vec{Z}}(\vec{z}) = \frac{1}{(2\pi)^{k/2}\sqrt{\det(\mathbf{K}_{\vec{Z}})|}} \exp\left[-\frac{1}{2}\vec{z}^{\mathsf{T}}\mathbf{K}_{\vec{Z}}^{-1}\vec{z}\right]$**.**

- $f_{\vec{Z}}(\vec{z}) = \prod_{j=1}^{k} \frac{1}{\sqrt{2\pi\lambda_j}} \exp\left[\frac{-|\langle\vec{z},\vec{q}_j\rangle|^2}{2\lambda_j}\right]$ **for** $\{\vec{q}_j\}$ **orthonormal,** $\{\lambda_j\}$ **positive.**

**$Z(t)$ is a Gaussian process if $Z(t_1), \ldots, Z(t_k)$ is jointly Gauss for all $k$ and $t_1, \ldots, t_k$.**

$$Z(t) = \sum_j Z_j \phi_j(t)$$

**where $\{Z_j, j \geq 1\}$ are stat. ind. Gauss and $\{\phi_j(ty); j \geq 1\}$ are orthonormal forms a large enough class of Gaussian random processes for the problems of interest.**

**If $\sum_j \sigma_j^2 < \infty$, the sample functions are $\mathcal{L}_2$.**

**If $Z(t)$ is a ZM Gaussian process and $g_1(t), \dots, g_k(t)$ are $\mathcal{L}_2$, then linear functions $V_j = \int Z(t) g_j(t)\, dt$ are jointly Gaussian.**

$$Z(t) \longrightarrow \boxed{h(t)} \longrightarrow V(\tau)$$

$$V(\tau) \;=\; \int_{-\infty}^{\infty} Z(t) h(\tau - t)\, dt$$

**For each $\tau$, this is a linear functional. $V(t)$ is a Gaussian process. It is stationary if $Z(t)$ is.**

$$\mathsf{K}_{\vec{V}}(r, s) \;=\; \int \int h(r-t) \mathsf{K}_{\vec{Z}}(t, \tau) h(s-\tau)\, dt\, d\tau$$

$\{Z(t); t \in \mathbb{R}\}$ **is stationary if** $Z(t_1), \dots, Z(t_k)$ **and** $Z(t_1+\tau), \dots, Z(t_k+\tau)$ **have same distribution for all** $\tau$**, all** $k$**, and all** $t_1, \dots, t_k$**.**

**Stationary implies that**

$$\mathsf{K}_{\vec{Z}}(t_1, t_2) = \mathsf{K}_{\vec{Z}}(t_1 - t_2, 0) = \tilde{\mathsf{K}}_{\vec{Z}}(t_1 - t_2). \qquad (3)$$

**A process is WSS if (3) holds. For Gaussian process, (3) implies stationarity.**

$\tilde{\mathsf{K}}_{\vec{Z}}(t)$ **is real and symmetric; Fourier transform** $S_{\vec{Z}}(f)$ **is real and symmetric..**

**An important example is the sampling expansion,**

$$V(t) = \sum_k V_k \,\mathsf{sinc}(\frac{t - kT}{T})$$

**Example 1: Let the $V_k$ be iid binary antipodal. Then $v(t)$ is WSS, but not stationary.**

**Example 2: Let the $V_k$ be iid zero-mean Gauss. then $V(t)$ is WSS and stationary (and zero-mean Gaussian).**

**For $Z(t)$ WSS and $g_1(t), \dots, g_k(t)$ $\mathcal{L}_2$,**

$$V_j = \int Z(t) g_j(t)\, dt.$$

$$\begin{aligned} \mathsf{E}[V_i V_j] &= \int_{t-\infty}^{\infty} g_i(t) \tilde{\mathsf{K}}_{\vec{Z}}(t-\tau) g_j(\tau)\, dt\, d\tau \\ &= \int \hat{g}_i(f) S_{\vec{Z}}(f) \hat{g}_j^*(f)\, df \end{aligned}$$

**If $\hat{g}_i(f)$ and $\hat{g}_j(f)$ do not overlap in frequency, then $\mathsf{E}[V_i V_j] = 0$. For $i = j$ and $g_i$ orthonormal, and $S_{\vec{Z}}(f)$ constant over $\hat{g}_i(f) \neq 0$,**

$$\mathsf{E}[|V_i|^2] = S_{\vec{Z}}(f)$$

**This means that $S(f)$ is the noise power per degree of freedom at frequency $f$.**

## LINEAR FILTERING OF PROCESSES

$$\{Z(t);\, t \in \Re\} \longrightarrow \boxed{h(t)} \longrightarrow \{V(\tau);\, \tau \in \Re\}$$

$$V(\tau) = \int_{-\infty}^{\infty} Z(t) h(\tau - t)\, dt.$$

$$S_{\vec{V}}(f) \;=\; S_{\vec{Z}}(f)\, |\hat{h}(f)|^2$$

**We can create a process of arbitrary spectral density in a band by filtering the IID sinc process in that band.**

White noise is noise that is stationary over a large enough frequency band to include all frequency intervals of interest, i.e., $S_{\vec{Z}}(f)$ is constant in $f$ over all frequencies of interest.

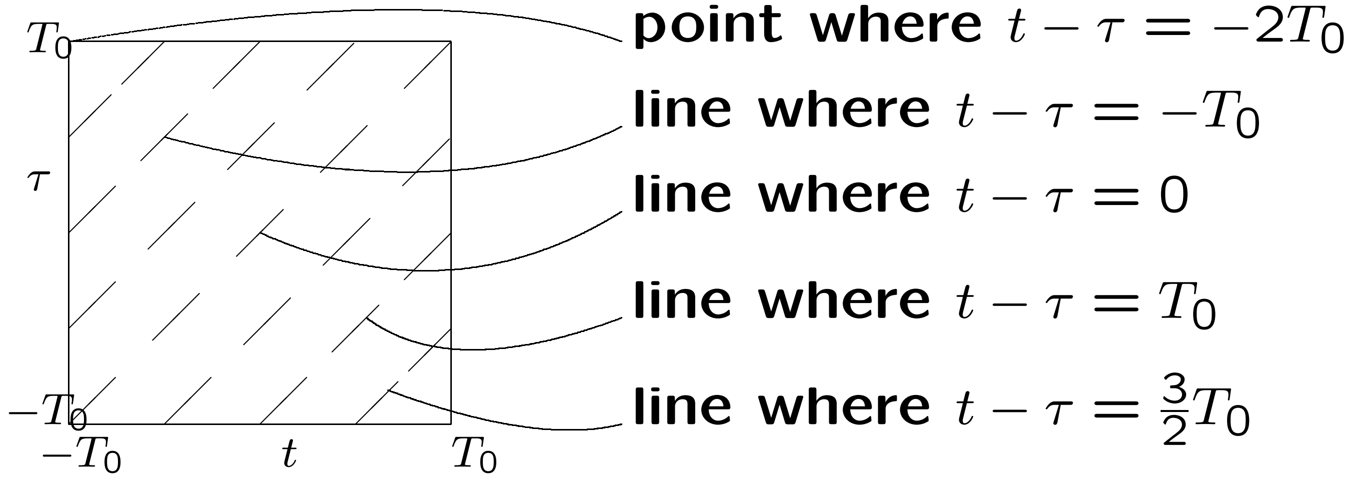
Within that frequency interval, $S_{\vec{Z}}(f)$ can be taken for many purposes as $N_0/2$ and $\tilde{\mathsf{K}}_{\vec{Z}}(t) = \frac{N_0}{2}\delta(t)$.

It is important to always be aware that this doesn't apply for frequencies outside the band of interest and doesn't make physical sense over all frequencies.

If the process is also Gaussian, it is called white Gaussian noise (WGN).

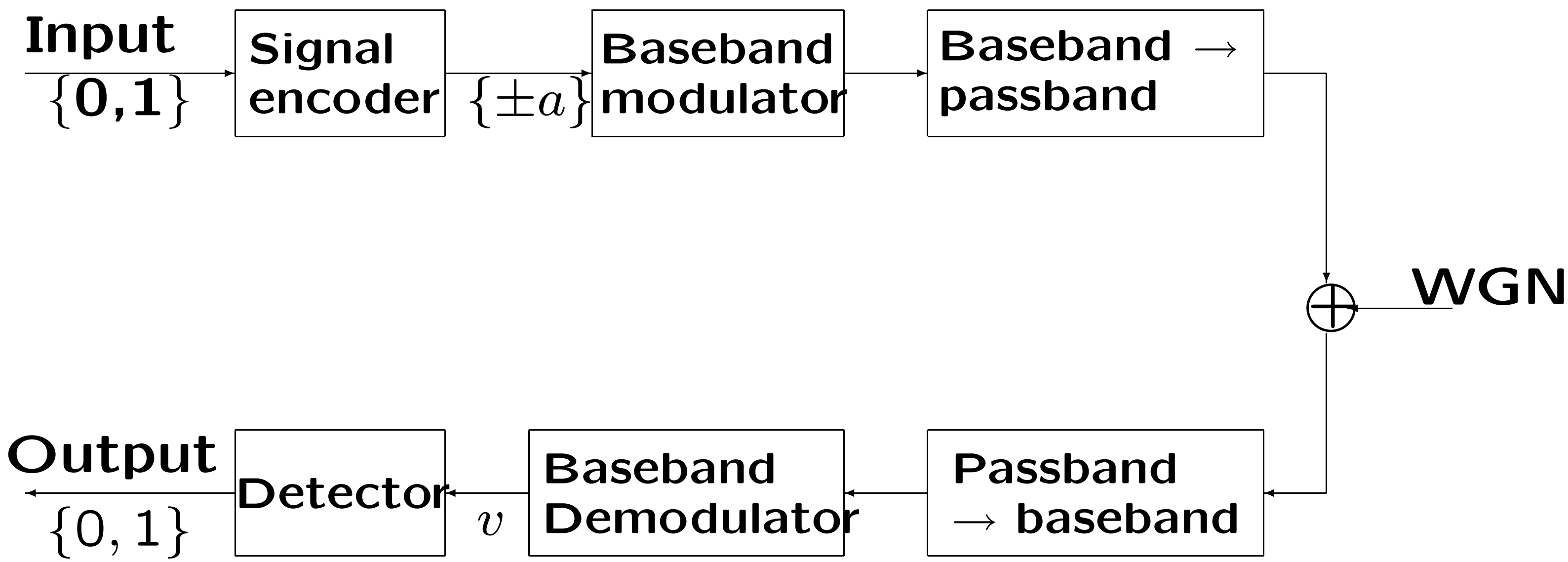
**Definition: A zero-mean random process is effectively stationary (effectively WSS) within $[-T_0, T_0]$ if the joint probability assignment (covariance matrix) for $t_1, \ldots, t_k$ is the same as that for $t_1+\tau, t_2+\tau, \ldots, t_k+\tau$ whenever $t_1, \ldots, t_k$ and $t_1+\tau, t_2+\tau, \ldots, t_k+\tau$ are all contained in the interval $[-T_0, T_0]$.**

**A process is effectively WSS within $[-T_0, T_0]$ if $\mathsf{K}_{\vec{Z}}(t, \tau) = \tilde{\mathsf{K}}_{\vec{Z}}(t-\tau)$ for $t, \tau \in [-T_0, T_0]$.**




**Note that** $\tilde{\mathbf{K}}_{\vec{Z}}(t-\tau)$ **for** $t,\tau\in[-T_0,T_0]$ **is defined in the interval** $[-2T_0,2T_0]$**.**

# DETECTION




**A detector observes a sample value of a rv** $V$ **(or vector, or process) and guesses the value of another rv,** $H$ **with values** $0, 1, \ldots, m-1$**.**

**Synonyms: hypothesis testing, decision making, decoding.**

**We assume that the detector uses a known probability model.**

**We assume the detector is designed to maximize the probability of guessing correctly (i.e., to minimize the probability of error).**

**Let $H$ be the rv to be detected (guessed) and $V$ the rv to be observed.**

**The experiment is performed, $V = v$ is observed and $H = j$, is not observed; the detector chooses $\widehat{H}(v) = i$, and an error occurs if $i \neq j$.**

**In principle, the problem is simple.**

**Given $V = v$, we calculate $p_{H|V}(j \mid v)$ for each $j$, $0 \le j \le m-1$.**

**This <u>is</u> the probability that $j$ is the correctcon- ditional on $v$. The MAP (maximum a posteriori probability) rule is: choose $\widehat{H}(v)$ to be that $j$ for which $p_{H|V}(j \mid v)$ is maximized.**

$$\widehat{H}(v) = \arg\max_j [p_{H|V}(j \mid v)] \qquad \textbf{(MAP rule)},$$

**The probability of being correct is $p_{H|V}(j \mid v)$ for that $j$. Averaging over $v$, we get the overall probability of being correct.**

# BINARY DETECTION

$H$ takes the values 0 or 1 with probabilities $p_0$ and $p_1$. We assume initially that only one binary digit is being sent rather than a sequence.

Assume initially that the demodulator converts the received waveform into a sample value of a rv with a probability density.

Usually the conditional densities $f_{V|H}(v \,|\, j)$, $j \in \{0, 1\}$ can be found.

These are called likelihoods. The marginal densiity of $V$ is then

$$f_V(v) = p_0 f_{V|H}(v \,|\, 0) + p_1 f_{V|H}(v \,|\, 1)$$

$$p_{H|V}(j\,|\,v) = \frac{p_j f_{V|H}(v\,|\,j)}{f_V(v)}.$$

**The MAP decision rule is**

$$\frac{p_0 f_{V|H}(v\,|\,0)}{f_V(v)} \quad \mathop{\gtreqless}_{\hat{H}=1}^{\hat{H}=0} \quad \frac{p_1 f_{V|H}(v\,|\,1)}{f_V(v)}.$$

$$\Lambda(v) = \frac{f_{V|H}(v\,|\,0)}{f_{V|H}(v\,|\,1)} \quad \mathop{\gtreqless}_{\hat{H}=1}^{\hat{H}=0} \quad \frac{p_1}{p_0} = \eta.$$