Fundamentals of probability. 6.436/15.085

LECTURE 24

Markov chains II. Mean recurrence times.

24.1. Markov chains with a single recurrence class

Recall the relations $\rightarrow, \leftrightarrow$ introduced in the previous lecture for the class of finite state Markov chains. Recall that we defined a state i to be recurrent if whenever $i \rightarrow j$ we also have $j \rightarrow i$, namely $i \leftrightarrow j$. We have observed that $\leftrightarrow$ is an equivalency relation, so that set of recurrent states is partitioned into equivalency classes $R_1, \ldots, R_r$. The remaining states $\mathcal{T}$ are transient.

Lemma 24.1. *For every $\forall i \in R, j \notin R$ we must have $p_{i,j} = 0$.*

This means that once the chain is in some recurrent class R it stays there forever.

Proof. The proof is simple: $p_{i,j} > 0$ implies $i \rightarrow j$. Since i is recurrent then also $j \rightarrow i$ implying $j \in R$ - contradiction. □

Introduce the following basic random quantities. Given states i, j let

$$T_i = \min\{n \geq 1 : X_n = i | X_0 = i\}.$$

In case no such n exists, we set $T_i = \infty$. (Thus the range of T_i is $\mathbb{N} \cup \{\infty\}$.) The quantity is called the *the first passage time*.

Lemma 24.2. *For every state $i \in \mathcal{T}$, $\mathbb{P}(X_n = i, \text{ i.o.}) = 0$. Namely, almost surely, after some finite time n_0, the chain will never return to i. In addition $\mathbb{E}[T_i] = \infty$.*

Proof. By definition there exists a state $j \notin \mathcal{T}$ such that $i \rightarrow j$, $j \nrightarrow i$. It then follows that $\mathbb{P}(T_i = \infty) > 0$ implying $\mathbb{E}[T_i] = \infty$. Now, let us establish the first part.

Let $I_{i,m}$ be the indicator of the event that the M.c. returned to state i at least m times. Notice that $\mathbb{P}(I_{i,1}) = \mathbb{P}(T_i < \infty) < 1$. Also by M.c. property we have $\mathbb{P}(I_{i,m} | I_{i,m-1}) = \mathbb{P}(T_i < \infty)$, as conditioning that at some point the M.c. returned to state i $m-1$ times does not impact its likelihood to return to this state again. Also notice $I_{i,m} \subset I_{i,m-1}$. Thus $\mathbb{P}(I_{i,m}) = \mathbb{P}(I_{i,m} | I_{i,m-1})\mathbb{P}(I_{i,m-1}) = \mathbb{P}(T_i < \infty)\mathbb{P}(I_{i,m-1}) = \cdots = \mathbb{P}^m(T_i < \infty)$. Since $\mathbb{P}(T_i < \infty) < 1$, then by continuity of probability property we obtain $\mathbb{P}(\cap_m I_{i,m}) = \lim_{m\rightarrow\infty} \mathbb{P}(I_{i,m}) = \lim_{m\rightarrow\infty} \mathbb{P}^m(T_i < \infty) = 0$. Notice that the event $\cap_m I_{i,m}$ is precisely the event $X_n = i$, i.o. □

We now focus on the family of Markov chains with only one recurrent class. Namely $\mathcal{X} = \mathcal{T} \cup R$. If in addition $\mathcal{T} = \emptyset$, then such a M.c. is called *irreducible*.

Exercise 1. Show that $\mathcal{T} \neq \mathcal{X}$. Namely, in every finite state M.c. there exists at least one recurrent state.

Exercise 2. Let $i \in \mathcal{T}$ and let π be an arbitrary stationary distribution. Establish that $\pi_i = 0$.

Exercise 3. Suppose M.c. has one recurrent class R. Show that for every $i \in R$ $\mathbb{P}(X_n = i, \text{ i.o.}) = 1$. Moreover, show that there exists $0 < q < 1$ and $C > 0$ such that $\mathbb{P}(T_i > t) \leq Cq^t$ for all $t \geq 0$. As a result, show that $\mathbb{E}[T_i] < \infty$.

Let $\mu_i = \mathbb{E}[T_i]$, possibly with $\mu_i = \infty$. This is called mean recurrence time of the state i.

24.2. Uniqueness of the stationary distribution

We now establish a fundamental result on M.c. with a single recurrence class.

Theorem 24.3. *A finite state M.c. with a single recurrence class has a unique stationary distribution π, which is given as $\pi_i = \frac{1}{\mu_i}$ for all states i. Specifically, $\pi_i > 0$ iff the state i is recurrent.*

Proof. Let P be the transition matrix of the chain. We let the state space be $\mathcal{X} = \{1, \ldots, N\}$. We fix an arbitrary recurrent state k. We know that one exists by Exercise 1. Let N_i be the number of visits to state i between two successive visits to state k. In case $i = k$, the last visit is counted but the initial is not. Namely, in the special case $i = k$ the number of visits is 1 with probability one. and let $\rho_i(k) = \mathbb{E}[N_i]$. Consider the event $\{X_n = i, T_k \geq n\}$ and consider the indicator function $\sum_{n\geq 1} I_{X_n=i,T_k\geq n} = \sum_{1\leq n\leq T_k} I_{X_n=i}$. Notice that this sum is precisely N_i. Namely,

$$\rho_i(k) = \sum_{n\geq 1} \mathbb{P}(X_n = i, T_k \geq n | X_0 = k). \tag{24.4}$$

Then using the formula $\mathbb{E}[Z] = \sum_{n\geq 1} \mathbb{P}(Z \geq n)$ for integer valued r.v., we obtain

$$\sum_i \rho_i(k) = \sum_{n\geq 1} \mathbb{P}(T_k \geq n | X_0 = k) = \mathbb{E}[T_k] = \mu_k. \tag{24.5}$$

Since k is recurrent, then by Exercise 3, $\mu_k < \infty$ implying $\rho_i(k) < \infty$. We let $\rho(k)$ denote the vector with components $\rho_i(k)$.

Lemma 24.6. *$\rho(k)$ satisfies $\rho^T(k) = \rho^T(k)P$. In particular, for every recurrent state k, $\pi_i = \frac{\rho_i(k)}{\mu_k}, 1 \leq i \leq N$ defines a stationary distribution.*

Proof. The second part follows from (24.5) and the fact that $\mu_k < \infty$. Now we prove the first part. We have for every $n \geq 2$

$$\mathbb{P}(X_n = i, T_k \geq n | X_0 = k) = \sum_{j\neq k} \mathbb{P}(X_n = i, X_{n-1} = j, T_k \geq n | X_0 = k) \tag{24.7}$$

$$= \sum_{j\neq k} \mathbb{P}(X_{n-1} = j, T_k \geq n-1 | X_0 = k) p_{j,i} \tag{24.8}$$

Observe that $\mathbb{P}(X_1 = i, T_k \geq 1 | X_0 = k) = p_{k,i}$. We now sum the (24.7) over n and apply it to (24.4) to obtain

$$\rho_i(k) = p_{k,i} + \sum_{j\neq k} \sum_{n\geq 2} \mathbb{P}(X_{n-1} = j, T_k \geq n-1 | X_0 = k) p_{j,i}$$

We recognize $\sum_{n\geq 2}\mathbb{P}(X_{n-1}=j, T_k\geq n-1|X_0=k)$ as $\rho_j(k)$. Using $\rho_k(k)=1$ we obtain

$$\rho_i(k)=\rho_k(k)p_{k,i}+\sum_{j\neq k}\rho_j(k)p_{j,i}=\sum_j \rho_j(k)p_{j,i}$$

which is in vector form precisely $\rho^T(k)=\rho^T(k)P$. □

We now return to the proof of the theorem. Let π denote an *arbitrary* stationary distribution of our M.c. We know one exists by Lemma 24.6 and, independently by our linear programming based proof. By Exercise 2 we already know that $\pi_i=1/\mu_i=0$ for every transient state i.

We now show that $\pi_k=1/\mu_k$ for every recurrent state k. Fix an arbitrary stationary distribution π. Assume that at time zero we start with distribution π. Namely $\mathbb{P}(X_0=i)=\pi_i$ for all i. Of course this implies that $\mathbb{P}(X_n=i)$ is also π_i for all n. On the other hand, fix any recurrent state k and consider

$$\mu_k\pi_k=\mathbb{E}[T_k|X_0=k]\mathbb{P}(X_0=k)=\sum_{n\geq 1}\mathbb{P}(T_k\geq n|X_0=k)\mathbb{P}(X_0=k)=\sum_{n\geq 1}\mathbb{P}(T_k\geq n, X_0=k)$$

On the other hand $\mathbb{P}(T_k\geq 1, X_0=k)=\mathbb{P}(X_0=k)$ and for $n\geq 2$

$$\begin{aligned}\mathbb{P}(T_k\geq n, X_0=k)&=\mathbb{P}(X_0=k, X_j\neq k, 1\leq j\leq n-1)\\&=\mathbb{P}(X_j\neq k, 1\leq j\leq n-1)-\mathbb{P}(X_j\neq k, 0\leq j\leq n-1)\\&\overset{(*)}{=}\mathbb{P}(X_j\neq k, 0\leq j\leq n-2)-\mathbb{P}(X_j\neq k, 0\leq j\leq n-1)\\&=a_{n-2}-a_{n-1},\end{aligned}$$

where $a_n=\mathbb{P}(X_j\neq k, 0\leq j\leq n)$ and (*) follows from stationarity of π. Now $a_0=\mathbb{P}(X_0\neq k)$. Putting together, we obtain

$$\begin{aligned}\mu_k\pi_k&=\mathbb{P}(X_0=k)+\sum_{n\geq 2}(a_{n-2}-a_{n-1})\\&=\mathbb{P}(X_0=k)+\mathbb{P}(X_0\neq k)-\lim_n a_n\\&=1-\lim_n a_n\end{aligned}$$

But by continuity of probabilities $\lim_n a_n=\mathbb{P}(X_n\neq k,\forall n)$. By Exercise 3, the state k, being recurrent is visited infinitely often with probability one. We conclude that $\lim_n a_n=0$, which gives $\mu_k\pi_k=1$, implying that π_k is uniquely defined as $1/\mu_k$. □

24.3. Ergodic theorem

Let $N_i(t)$ denote the number of times the state i is visited during the times $0,1,\ldots,t$. What can be said about the behavior of $N_i(t)/t$ when t is large? The answer turns out to be very simple: it is π_i. These type of results are called *ergodic* properties, as they show how the time average of the system, namely $N_i(t)/t$ relates to the spatial average, namely π_i.

Theorem 24.9. *For arbitrary starting state* $X_0=k$ *and for every state* i*,* $\lim_{t\to\infty}\frac{N_i(t)}{t}=\pi_i$ *almost surely. Also* $\lim_{t\to\infty}\frac{\mathbb{E}[N_i(t)]}{t}=\pi_i$.

Proof. Suppose $X_0=k$. If i is a transient state, then, as we have established, almost surely after some finite time, the chain will never enter i, meaning $\lim_t N_i(t)/t=0$ almost surely. Since also $\pi_i = 0$, then we have established the required equality for the case when i is a transient state.

Suppose now i is a recurrent state. Let $T_1, T_2, T_3, \ldots$ denote the time of successive visits to i. Then the sequence $T_n, n \geq 2$ is i.i.d. Also T_1 is independent from the rest of the sequence, although it distribution is different from the one of $T_m, m \geq 2$ since we have started the chain from k which is in general different from i. By the definition of $N_i(t)$ we have

$$\sum_{1 \leq m \leq N_i(t)} T_m \leq t < \sum_{1 \leq m \leq N_i(t)+1} T_m$$

from which we obtain

$$\frac{\sum_{1 \leq m \leq N_i(t)} T_m}{N_i(t)} \leq \frac{t}{N_i(t)} < \frac{\sum_{1 \leq m \leq N_i(t)+1} T_m}{N_i(t)+1} \frac{N_i(t)+1}{N_i(t)}. \tag{24.10}$$

We know from Exercise 3 that $\mathbb{E}[T_m] < \infty, m \geq 2$. Using a similar approach it can be shown that $\mathbb{E}[T_1] < \infty$, in particular $T_1 < \infty$ a.s. Applying SLLN we have that almost surely

$$\lim_{n \to \infty} \frac{\sum_{2 \leq m \leq n} T_m}{n} = \lim_{n \to \infty} \frac{\sum_{2 \leq m \leq n} T_m}{n-1} \frac{n-1}{n} = \mathbb{E}[T_2]$$

which further implies

$$\lim_{n \to \infty} \frac{\sum_{1 \leq m \leq n} T_m}{n} = \lim_{n \to \infty} \frac{\sum_{2 \leq m \leq n} T_m}{n} + \lim_{n \to \infty} \frac{T_1}{n} = \mathbb{E}[T_2]$$

almost surely. □

Since i is a recurrent state then by Exercise 3, $N_i(t) \to \infty$ almost surely as $t \to \infty$. Combining the preceding identity with (24.10) we obtain

$$\lim_{t \to \infty} \frac{t}{N_i(t)} = \mathbb{E}[T_2] = \mu_i,$$

from which we obtain $\lim_t N_i(t)/t = \mu_i^{-1} = \pi_i$ almost surely.

To establish the convergence in expectation, notice that $N_i(t) \leq t$ almost surely, implying $N_i(t)/t \leq 1$. Applying bounded convergence theorem, we obtain that $\lim_t \mathbb{E}[N_i(t)]/t = \pi_i$, and the proof is complete.

24.4. Markov chains with multiple recurrence classes

How does the theory extend to the case when the M.c. has several recurrence classes $R_1, \ldots, R_r$? The summary of the theory is as follows (the proofs are very similar to the case of single recurrent class case and is omitted). It turns out that such a M.c. chain possesses r stationary distributions $\pi^i = (\pi^i_1, \ldots, \pi^i_N), 1 \leq i \leq r$, each "concentrating" on the class R_i. Namely for each i and each state $k \notin R_i$ we have $\pi^i_k = 0$. The i-th stationary distribution is described by $\pi^i_k = 1/\mu_k$ for all $k \in R_i$ and where μ_k is the mean return time from state $k \in R_j$ into itself. Intuitively, the stationary distribution π^i corresponds to the case when the M.c. "lives" entirely in the class R_i. One can prove that the family of all of the stationary distributions of such a M.c. can be obtained by taking all possible convex combinations of $\pi^i, 1 \leq i \leq r$, but we omit the proof. (Show that a convex combination of stationary distributions is a stationary distribution).

24.5. References

- Sections 6.3-6.4 [**1**].

BIBLIOGRAPHY

1. G. R. Grimmett and D. R. Stirzaker, *Probability and random processes*, Oxford University Press, 2005.