Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science

6.341: Discrete-Time Signal Processing

Fall 2005

Problem Set 4 Solutions

Issued: Tuesday, October 4, 2005

Problem 4.1 (OSB 6.10)

(a)

$$w[n] = \frac{1}{2}y[n] + x[n]$$

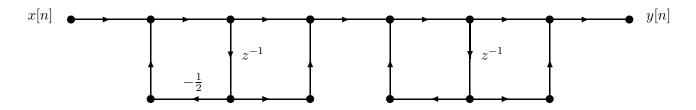$$v[n] = \frac{1}{2}y[n] + 2x[n] + w[n-1]$$

$$y[n] = x[n] + v[n-1]$$

(b) We can take the z-transforms of the difference equations in part (a) and obtain the system function algebraically. Alternatively, we can recognize the flow graph in OSB Figure P6.10 - 1 as a transposed direct form II structure. Either way, we find:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 + 2z^{-1} + z^{-2}}{1 - \frac{1}{2}z^{-1} - \frac{1}{2}z^{-2}}$$

We can factor the numerator and the denominator into:

$$H(z) = \frac{\left(1 + z^{-1}\right)\left(1 + z^{-1}\right)}{\left(1 + \frac{1}{2}z^{-1}\right)\left(1 - z^{-1}\right)}$$

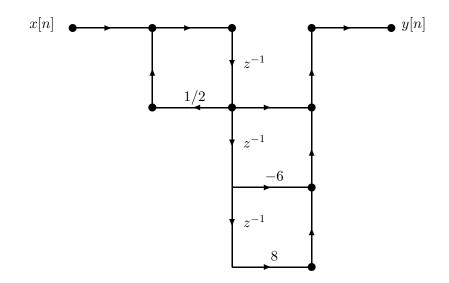The factorization suggests the following cascade form:

(c) The system function has poles at $z = -\frac{1}{2}$ and $z = 1$. Since the second pole is on the unit circle, the system is not stable.
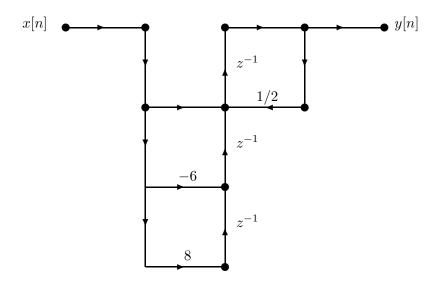
Problem 4.2 (OSB 6.11)

(a) $H(z)$ can be expanded as:

$$H(z) = \frac{z^{-1} - 6z^{-2} + 8z^{-3}}{1 - \frac{1}{2}z^{-1}}$$

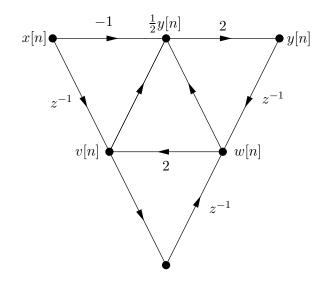$H(z)$ can then be translated into the following direct form II flow graph:

(b) To get the transposed form, we reverse the arrows and exchange the input and the output. The graph can then be redrawn as:

Problem 4.3 (OSB 6.12)

OSB Figure P6.12-1 is redrawn below with some intermediate nodes labelled. We notice that the upper centre node is $\frac{1}{2}y[n]$ and that the lowest node is also $v[n]$.

Writing the z-transform equations for $v[n]$, $w[n]$ and $y[n]$:

$$V(z) = z^{-1}X(z) + 2W(z), \tag{1a}$$

$$W(z) = z^{-1}Y(z) + z^{-1}V(z), \tag{1b}$$

$$Y(z) = -2X(z) + 2V(z) + 2W(z). \tag{1c}$$

Using equations (1a) and (1b), we express $V(z)$ and $W(z)$ in terms of $X(z)$ and $Y(z)$:

$$V(z) = \frac{z^{-1}X(z) + 2z^{-1}Y(z)}{1 - 2z^{-1}}, \tag{2a}$$

$$W(z) = \frac{z^{-2}X(z) + z^{-1}Y(z)}{1 - 2z^{-1}}. \tag{2b}$$

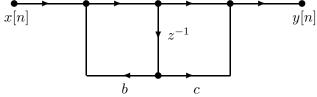Substituting equations (2a) and (2b) into (1c), we arrive at the difference equation for the system.

$$(1 - 8z^{-1})Y(z) = (-2 + 6z^{-1} + 2z^{-2})X(z),$$

$$y[n] - 8y[n-1] = -2x[n] + 6x[n-1] + 2x[n-2].$$
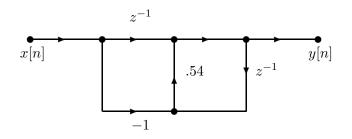
Problem 4.4 (OSB 6.33)

(a) The system function for the flowgraph below is:

$$H(z) = d\,\frac{cz^{-1} + 1}{1 - bz^{-1}} = cd\,\frac{z^{-1} + \frac{1}{c}}{1 - bz^{-1}}$$

Comparing with $H(z)$ as given,

$b = 0.54$, $1/c = -0.54$, and $cd = 1$, so $c = -1.852$ and $d = -0.54$.

(b) $H(z)$ has a zero at $-c$ and a pole at b. The system is allpass when $-c = 1/b^*$. With quantized coefficients $\hat{b}$, $\hat{c}$, and $\hat{d}$, $\hat{b}^*\hat{c} \neq -1$ in general, so the resulting system would not be allpass.

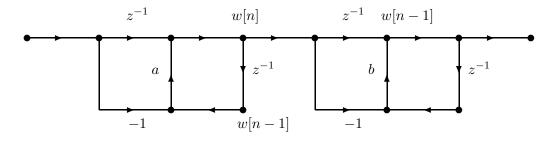(c) The flowgraph is shown below.

(d) **Yes.** There is only one "0.54" to quantize and its value determines the position of both the pole and zero. The structure in part (c) ensures that the pole and zero are always conjugate reciprocals regardless of the value of the multiplier.
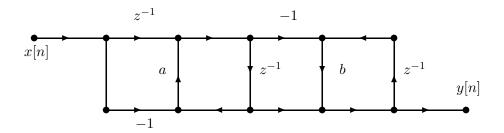
(e)

$$H(z) = \left(\frac{z^{-1} - a}{1 - az^{-1}}\right)\left(\frac{z^{-1} - b}{1 - bz^{-1}}\right)$$

Cascading two sections as in (c) gives

The second delay in the first section and the first delay in the second section both output $w[n-1]$. We can turn the second section upside-down and reuse the second delay in the first section.
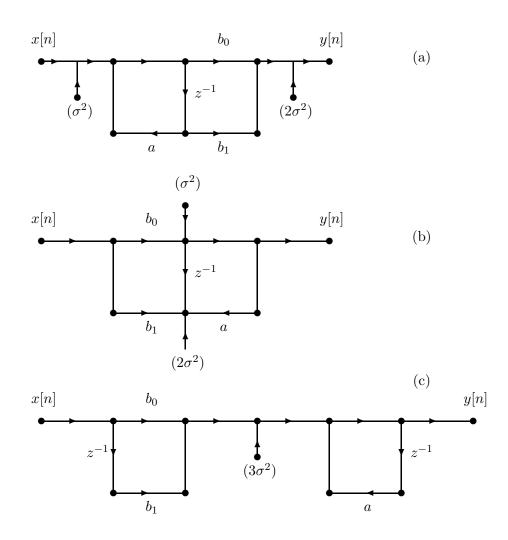

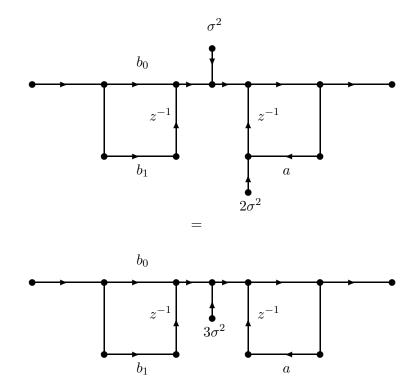
(f) Yes, same reason as in part (d).

Problem 4.5 (OSB 6.42)

(a) Every time we multiply a signal value by a coefficient (both represented by $(B+1)$ bits), we must round the $(2B+1)$-bit product to $(B+1)$ bits. Thus, a round-off noise source should be injected into the network after every multiplication.

We assume as usual that each noise source is a white, WSS random process with average power σ_B^2 and is uncorrelated with the input and all other noise sources. The linear noise model for each system is drawn below:

(b) System (a) is clearly different from the other two in its placement of noise sources. In system (c), all of the noise goes through the pole network. If we split the delay in system (b) into two delays, we find:

In system (b), the upper noise source sees the same pole network as in (c). The two lower noise sources see the same pole network plus an additional delay (z^{-1}). However, delays do not affect the average power output. Hence systems (b) and (c) share the same output noise power.

(c) The output noise power is equal to the autocorrelation $\phi_{yy}[m]$ evaluated at $m = 0$. Denoting the impulse response seen by noise source i as $h_i[n]$ and using superposition,

$$
\begin{aligned}
\phi_{yy}[m] &= \sum_{i=1}^{3} \sigma_B^2 \delta[m] * h_i[m] * h_i[-m] \\
&= \sigma_B^2 \sum_{i=1}^{3} \sum_{k=-\infty}^{\infty} h_i[k] h_i[m+k] \\
\phi_{yy}[0] &= \sigma_B^2 \sum_{i=1}^{3} \sum_{k=-\infty}^{\infty} h_i^2[k].
\end{aligned}
$$

For network (c) (and (b)), all noise sources are filtered by:

$$
H(z) = \frac{1}{1 - az^{-1}},
$$

$$h[n] = a^n u[n].$$

$$\phi_{yy}[0] = 3\sigma_B^2 \sum_{n=0}^{\infty} (a^n)^2$$

$$= \frac{3\sigma_B^2}{1 - a^2},$$

For network (a), the noise source at the input is subjected to:

$$H(z) = \frac{b_0 + b_1 z^{-1}}{1 - az^{-1}},$$

$$h[n] = b_0 \delta[n] + \left(b_0 + \frac{b_1}{a} \right) a^n u[n-1].$$

The other two noise sources enter directly at the output.

$$\phi_{yy}[0] = 2\sigma_B^2 + \sigma_B^2 \sum_{n=0}^{\infty} h^2[n]$$

$$= 2\sigma_B^2 + \sigma_B^2 \left(b_0^2 + \left(b_0 + \frac{b_1}{a} \right)^2 \underbrace{\sum_{n=1}^{\infty} a^{2n}}_{\frac{a^2}{1-a^2}} \right)$$

$$= 2\sigma_B^2 + \sigma_B^2 \left(b_0^2 + \frac{(ab_0 + b_1)^2}{1 - a^2} \right)$$

$$= 2\sigma_B^2 + \sigma_B^2 \left(\frac{b_0^2 + 2ab_0 b_1 + b_1^2}{1 - a^2} \right).$$
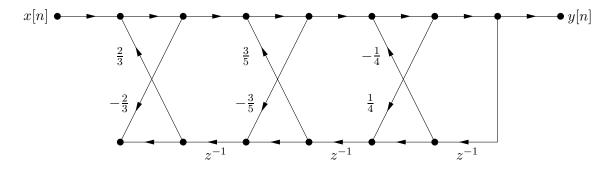
Problem 4.6

(a) The FIR lattice filter has three stages and is therefore third-order. The three k-parameters or reflection coefficients are:

$$k_1 = -\frac{1}{4}, \quad k_2 = \frac{3}{5}, \quad k_3 = \frac{2}{3}.$$

We start the recursion given by equations (8) in the lattice filter notes at $p + 1 = 1$ and proceed until we have found the coefficients $a_k^{(3)}, \quad k = 1, 2, 3$.

$$a_1^{(1)} = k_1 = -\frac{1}{4}$$

$$a_1^{(2)} = a_1^{(1)} - k_2 a_1^{(1)} = -\frac{1}{10}$$

$$a_2^{(2)} = k_2 = \frac{3}{5}$$

$$a_1^{(3)} = a_1^{(2)} - k_3 a_2^{(2)} = -\frac{1}{2}$$

$$a_2^{(3)} = a_2^{(2)} - k_3 a_1^{(2)} = \frac{2}{3}$$

$$a_3^{(3)} = k_3 = \frac{2}{3}$$

$$H(z) = 1 + \frac{1}{2}z^{-1} - \frac{2}{3}z^{-2} - \frac{2}{3}z^{-3}.$$

(b) Armed with the k-parameters from part (a), we refer to Figure 5 in the lattice filter notes in drawing the lattice structure for the all-pole filter $1/H(z)$:

(Note: $1/H(z)$ is stable because all of the reflection coefficients have magnitudes less than unity.)

Problem 4.7

The all-pole filter is fourth-order with coefficients:

$$a_1^{(4)} = -\frac{3}{2}, \quad a_2^{(4)} = 1, \quad a_3^{(4)} = -\frac{3}{4}, \quad a_4^{(4)} = -2.$$

We know immediately that $k_4 = a_4^{(4)} = -2$. To find the remaining reflection coefficients, we need to run the recursion in reverse and find the coefficients for successively lower order filters. Letting $M = 4$ in equation (11) of the lattice filter notes,

$$a_1^{(3)} = \frac{a_1^{(4)} + k_4 a_3^{(4)}}{1 - k_4^2} = 0$$

$$a_2^{(3)} = \frac{a_2^{(4)} + k_4 a_2^{(4)}}{1 - k_4^2} = \frac{1}{3}$$

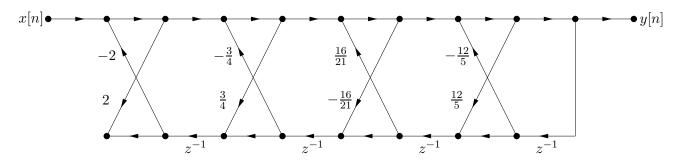$$a_3^{(3)} = \frac{a_3^{(4)} + k_4 a_1^{(4)}}{1 - k_4^2} = -\frac{3}{4}$$

We identify $k_3 = a_3^{(3)} = -\frac{3}{4}$ and proceed to $M = 3$:

$$a_1^{(2)} = \frac{a_1^{(3)} + k_3 a_2^{(3)}}{1 - k_3^2} = -\frac{4}{7}$$

$$a_2^{(2)} = \frac{a_2^{(3)} + k_3 a_1^{(3)}}{1 - k_3^2} = \frac{16}{21}$$

Thus $k_2 = a_2^{(2)} = \frac{16}{21}$. Finally,

$$a_1^{(1)} = \frac{(1 + k_2) a_1^{(2)}}{1 - k_2^2} = \frac{a_1^{(2)}}{1 - k_2} = -\frac{12}{5},$$

and $k_1 = -\frac{12}{5}$.

The lattice structure for $H(z)$ is shown below:

Since $|k_1| > 1$ and $|k_4| > 1$, the all-pole filter cannot be stable.

Problem 4.8 (OSB 5.68)

For parts (a) and (b) we need the following Fourier transform property:

$$h[-n] \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad H(e^{-j\omega})$$

$$\overset{\mathcal{F}}{\longleftrightarrow} \quad H^*(e^{j\omega}) \quad \text{if } h[n] \text{ is real}$$

(a) It is easier to work in the frequency domain in finding the overall impulse response $h_1[n]$.

$$G(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$$

$$
\begin{aligned}
R(e^{j\omega}) &= H(e^{j\omega})G(e^{-j\omega}) \\
&= H(e^{j\omega})H(e^{-j\omega})X(e^{-j\omega})
\end{aligned}
$$

$$S(e^{j\omega}) = R(e^{-j\omega}) = H(e^{-j\omega})H(e^{j\omega})X(e^{j\omega})$$
$$H_1(e^{j\omega}) = H(e^{j\omega})H(e^{-j\omega})$$

Hence,

$$h_1[n] = h[n] * h[-n]$$

Rewriting $H_1(e^{j\omega})$ for real $h[n]$:

$$H_1(e^{j\omega}) = H(e^{j\omega})H^*(e^{j\omega}) = |H(e^{j\omega})|^2$$

$H_1(e^{j\omega})$ is real, non-negative and therefore zero-phase. Its magnitude is given by:

$$|H_1(e^{j\omega})| = |H(e^{j\omega})|^2$$

(b)
$$G(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$$
$$R(e^{j\omega}) = H(e^{j\omega})X(e^{-j\omega})$$

$$
\begin{aligned}
Y(e^{j\omega}) &= G(e^{j\omega}) + R(e^{-j\omega}) \\
&= [H(e^{j\omega}) + H(e^{-j\omega})]X(e^{j\omega})
\end{aligned}
$$

$$H_2(e^{j\omega}) = H(e^{j\omega}) + H(e^{-j\omega})$$

Hence,

$$h_2[n] = h[n] + h[-n]$$

Rewriting $H_2(e^{j\omega})$ for real $h[n]$:

$$
\begin{aligned}
H_2(e^{j\omega}) &= H(e^{j\omega}) + H^*(e^{j\omega}) \\
&= 2Re\{H(e^{j\omega})\} \\
&= 2|H(e^{j\omega})|\cos(\angle H(e^{j\omega}))
\end{aligned}
$$

$H_2(e^{j\omega})$ is real and is zero-phase as long as we allow sign changes in its amplitude. Its magnitude is given by:

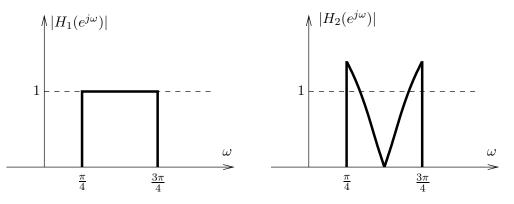$$|H_2(e^{j\omega})| = 2|H(e^{j\omega})||\cos(\angle H(e^{j\omega}))|$$

(c)

$$
\begin{aligned}
|H_1(e^{j\omega})| &= |H(e^{j\omega})|^2 \\
&= \begin{cases} 1, & \frac{\pi}{4} < |\omega| < \frac{3\pi}{4} \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
|H_2(e^{j\omega})| &= 2|H(e^{j\omega})||\cos(\angle H(e^{j\omega}))| \\
&= 2|H(e^{j\omega})||\cos\omega| \\
&= \begin{cases} 2|\cos\omega|, & \frac{\pi}{4} < |\omega| < \frac{3\pi}{4} \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

$|H_1(e^{j\omega})|$ and $|H_2(e^{j\omega})|$ are sketched below. Since we wanted a band-pass filter with a flat passband, $H_1(e^{j\omega})$ is clearly more desirable.

In general, method A $(H_1(e^{j\omega}))$ is preferable in achieving a zero-phase characteristic. If the desired magnitude response is $|H_1(e^{j\omega})|$, we could design $h[n]$ to have a magnitude response equal to $\sqrt{|H_1(e^{j\omega})|}$. Using method B, $H_2(e^{j\omega})$ has a magnitude response that follows the real part of $H(e^{j\omega})$, which very rarely corresponds to $|H(e^{j\omega})|$ given an arbitrary phase characteristic. Supposing the desired magnitude response is $|H_2(e^{j\omega})|$, it is difficult to design $h[n]$ so that its frequency response has a desired real part as compared to a desired magnitude.