# 6.302 Feedback Systems
Recitation 6: Steady-State Errors
Prof. Joel L. Dawson

A valid performance metric for any control system centers around the final error when the system reaches steady-state. That is, after all initial transients have died away, how close is the output to the desired output?

Look at the error in a feedback system.
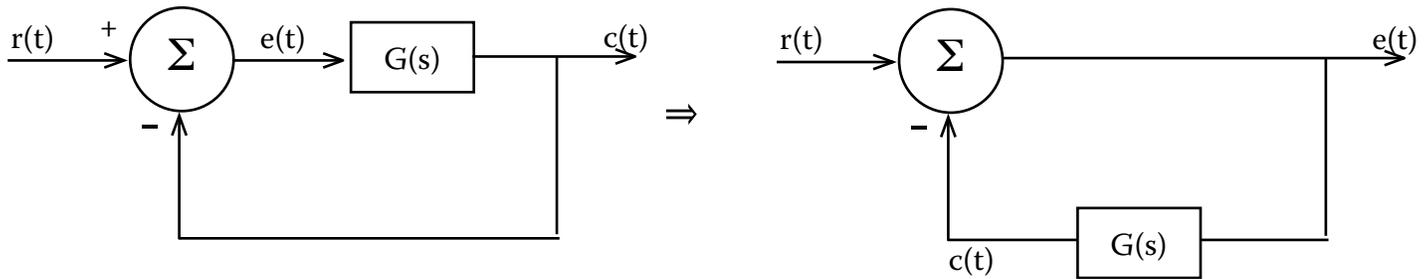


Use Black's Formula:

$$E(s) = \frac{1}{1 + G(s)}$$

For steady-state, we're interested in $\lim_{t\to\infty} e(t)$. Using the final value theorem, we can evaluate in the frequency domain using

$$\lim_{s\to 0} sF(s).$$

Now, what kinds of inputs might interest us when discussing steady state errors? Let's look back at our position control system from recitation 2. (We've added damping so that there's no oscillating.)
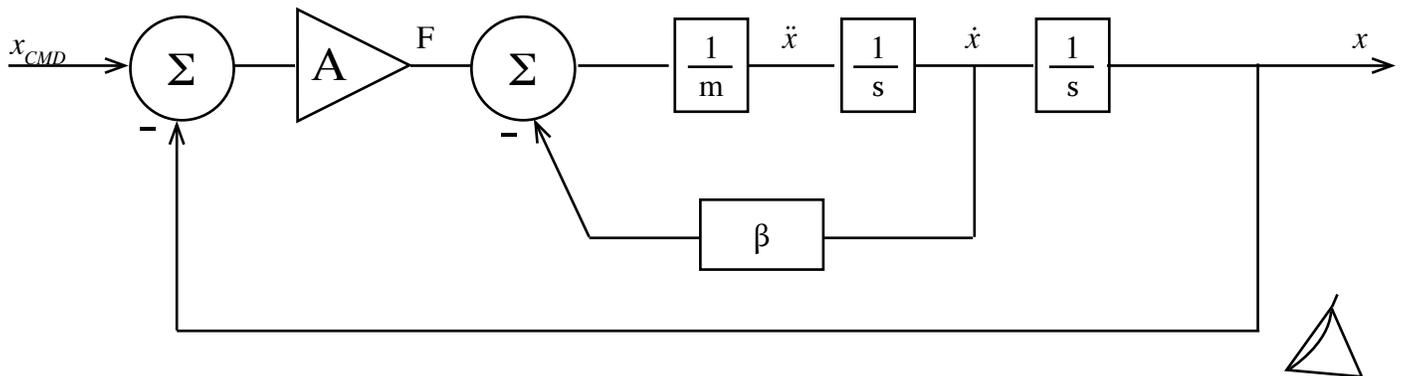
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If we wanted to command the mass to be moved, say, from the origin to $x_0$, our command might look something like:

$$x_{CMD}(t) = x_0 u(t) \qquad \left[ X_{CMD}(s) = \frac{x_0}{s} \right]$$
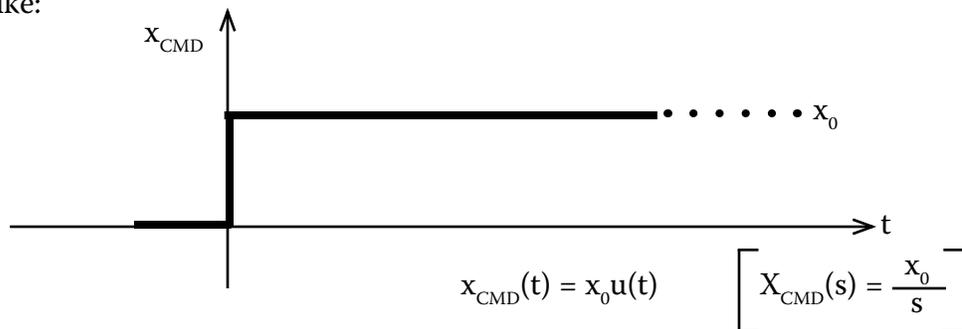
Suppose, instead, we wanted the block to be moved at a constant velocity. If we didn't have a velocity sensor handy, our command would take the form of a <u>ramp</u>:

$$x_{CMD} = v_0 t u(t) \qquad \left[ X_{CMD}(s) = \frac{v_0}{s^2} \right]$$
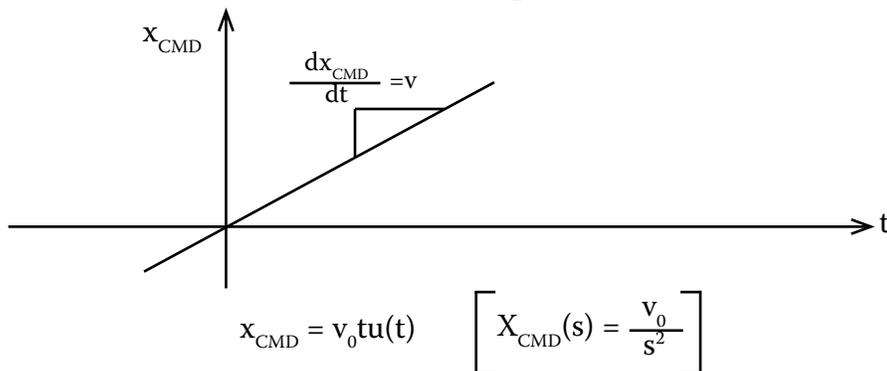
Or, we could be interested in moving with a constant acceleration. Lacking a good accelerometer, we could achieve this with our present control strategy using a parabolic input:

$$x_{CMD} = \tfrac{1}{2} a_0 t^2 u(t) \qquad \left[ X_{CMD}(s) = \frac{a_0}{s^3} \right]$$
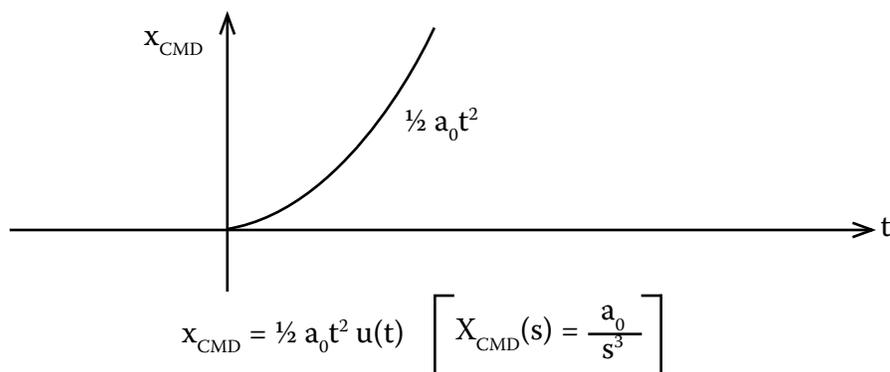
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It's not difficult to imagine more complicated inputs to a control system. Consider a rotating camera to track a running athelete:
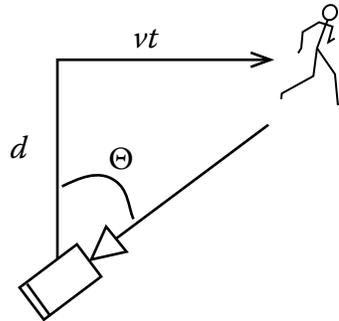


Figure by MIT OpenCourseWare.

We have:      $\text{TAN } \Theta = \dfrac{vt}{d}$ $\Rightarrow$ $\Theta(t) = \text{TAN}^{-1}\left(\dfrac{vt}{d}\right)$

Now we don't have a Laplace Transform for $\text{TAN}^{-1}$. However, we can write a Taylor Series for $\text{TAN}^{-1}$:

$$\Theta(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \dots$$

$$\Theta(s) = \frac{a_0}{s} + \frac{a_1}{s^2} + \frac{2a_2}{s^3} + \frac{6a_3}{s^4} + \dots$$

## DC Steady-State Error

Returning to our original system:
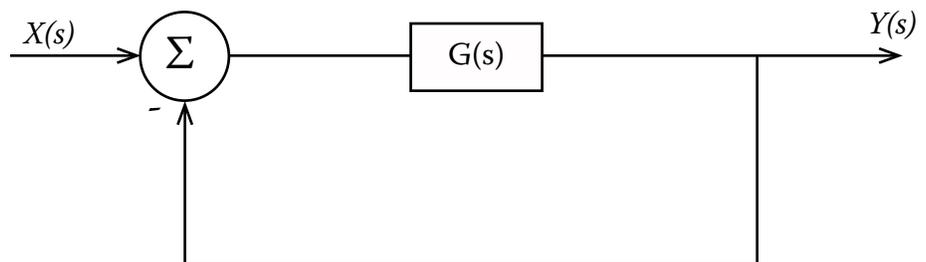


$$E = \frac{1}{1 + G(s)}$$

By "DC Steady-State Error," we mean "the final error in response to a step."

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Mathematically: $\quad \lim\limits_{t \to \infty} e(t) = \lim\limits_{s \to 0} s \left[ \dfrac{1}{s} \dfrac{1}{1 + G(s)} \right]$

CLASS EXERCISE:

Find a function G(s) for which the error in response to a step is zero.

It is instructive to work things out for different types of systems:
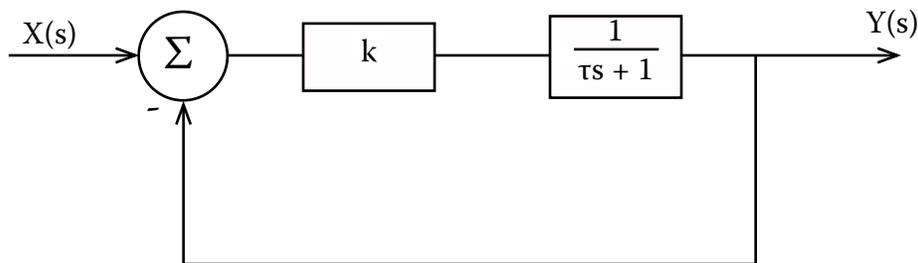


In response to a step: $\quad \lim\limits_{s \to 0} s \left[ \dfrac{1}{s} \cdot \dfrac{1}{1 + \dfrac{k}{\tau s + 1}} \right] = \dfrac{1}{1 + k}$

(The more we turn up the gain, the better we do.)

In response to a ramp: $\quad \lim\limits_{s \to 0} s \left[ \dfrac{1}{s^2} \cdot \dfrac{1}{1 + \dfrac{k}{\tau s + 1}} \right] = \infty$

(Error grows without bound.)

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What can we do about that ramp? Try:

X(s) → Σ → k(s) → $\frac{1}{s^2}$ → Y(s)
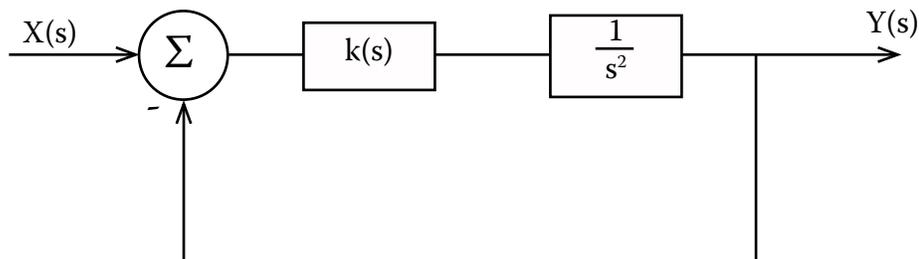
Where k(s) has no singularities at the origin. In other words, $|k(s)|_{s=0} = k_0$

Now the error for a ramp is:

$$\lim_{s \to 0} \left[ s \cdot \frac{1}{s^2} \; \frac{1}{1 + \frac{k(s)}{s^2}} \right]$$

$$\lim_{s \to 0} \left[ \frac{1}{s} \; \frac{s^2}{s^2 + k(s)} \right]$$

$$= 0$$

So for zero error in response to a step ($\frac{1}{s}$), we needed one integrator. For zero error in response to a ramp ($\frac{1}{s^2}$), we needed two integrators. Can we discern a pattern?

Let input be $X(S) = \frac{1}{s^M}$, and let forward path be k(s) $\frac{1}{s^N}$, where $k(0) = k_0 \Rightarrow$ apply final value theorem to generate expression:

$$\lim_{s \to 0} \; s \left[ \frac{1}{s^M} \; \frac{1}{1 + \frac{1}{s^N} k(s)} \right]$$

$$= \lim_{s \to 0} \left[ \frac{1}{s^{M-1}} \; \frac{s^N}{s^N + k_0} \right]$$

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$$\lim_{s \to 0} \left[ \frac{s^{N-m+1}}{s^N + k_0} \right] \overset{?}{=} 0$$

In order to be zero, we must have

$$N - M + 1 \geq 1$$

$$\boxed{N \geq M}$$

So in order to get zero steady-state error, we require:

| Input | # of poles @ origin |
|-------|--------------------|
| Step | 1 |
| Ramp | 2 |
| Parabolic | 3 |
| . | . |
| . | . |
| . | . |
| . | . |

Now, to re-emphasize a couple of points from lecture:

1)      Oscillation condition



Break loop here

see what comes back

inject $e^{j\omega_0 k}$

$+$

$-$   $e^{j\omega_0 k}$

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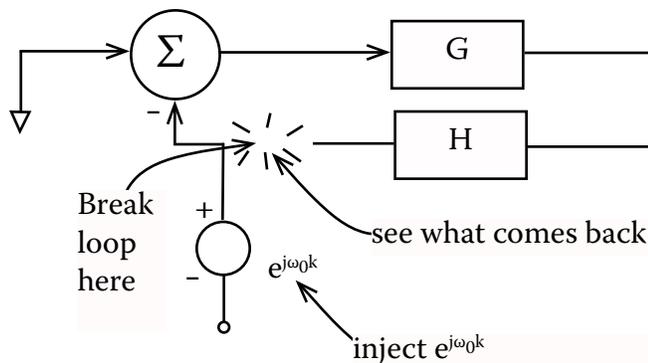
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If what comes back is just $e^{j\omega_0 t}$, that is,

$| L(j\omega_0) | = 1$ , and $\angle L(j\omega_0) = 180°$

We have an "oscillation condition," or a pole in the closed-loop transfer function:

$$\frac{G(s)}{1+L(s)}$$

↻ if $| L(j\omega_0) | = 1$ , and $\angle L(j\omega_0) = 180°$
denominator is zero.

### WARNING:  BEWARE OF FALSE INTUITION!

$\angle L(j\omega_0) = 180°$  AND  $| L(j\omega_0) | > 1$

DOES NOT necessarily imply instability/oscillations. Why? Because in this case, $1+ L(j\omega_0)$  is not zero, so $j\omega_0$ is not a pole of the closed-loop system. It is very common to get this wrong. Don't let it happen to you!!

Finally, a word about root locus. It answers the question: How do the closed-loop poles of the system move as I vary the gain k?



More on this next week.

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