

Massachusetts Institute of Technology

Department of Electrical Engineering and Computer Science  
6.245: MULTIVARIABLE CONTROL SYSTEMS

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## Problem Set 9 (due April 28, 2004) <sup>1</sup>

### Problem 9.1

The task of maximizing the probability of a correct outcome in a quantum detection experiment can be formulated as the convex optimization problem

$$\text{trace} \sum_{k=1}^m a_k X_k \rightarrow \max \text{ subject to } X_k = X_k' \geq 0, \sum_{k=1}^m X_k = I,$$

where  $a_i = a_i'$  are given  $n$ -by- $n$  symmetric matrices, and  $X_i = X_i'$  are symmetric matrices to be optimized.

- (a) Formulate a dual problem.
- (b) Describe the main components of a “no duality gap” statement: for a given  $y \in \mathbf{R}$ , define a convex set  $\Omega = \Omega_y$  and a vector  $g = g_y$ , such that zero is an interior point of  $\Omega_y$  for all  $y$  and  $g_y$  belongs to  $\Omega_y$  if and only if  $y$  is a strict upper bound for the maximum in the primal problem; explain how a feasible argument in the dual problem, for which the dual cost does not exceed  $y$ , can be derived from the parameters of a hyperplane separating  $\Omega_y$  from  $g_y$ .
- (c) Test your dual formulation by writing MATLAB code solving both primal and dual problems, and by checking that the maximum in the primal turns out to be close to the minimum the dual.

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<sup>1</sup>Version of April 21, 2004

### Problem 9.2

The open loop plant has scalar input  $u$ , scalar output  $y$ , and transfer function

$$P(s) = \frac{s - 1}{s(s - 2)}.$$

The objective is to design a proper LTI controller  $K$ , with inputs  $r$  (the reference signal) and  $y$ , and output  $u$ , to satisfy the following specifications:

- (i) the resulting closed loop system is stable;
- (ii) the absolute value  $|m_u(t)|$  of the closed loop unit step response  $m_u = m_u(t)$  from  $r$  to  $u$  never exceeds 10;
- (iii) the value  $m_y(t)$  of the closed loop unit step response  $m_y = m_y(t)$  from  $r$  to  $y$  never leaves the interval  $[-1, 2]$ .

The design objective is to minimize, subject to the specifications given above, the closed loop settling time, defined as the minimal  $T \geq 0$  such that

$$|1 - m_y(t)| \leq 0.1 \quad \forall t \geq T.$$

- (a) Derive Q-parameterization for all closed loop transfer matrices from  $r$  to  $z = [y; u]$  achievable by a stabilizing controller;
- (b) Explain how linear programming and the Q-parameterization can be used to find lower bounds  $T_0$  of arbitrary relative accuracy for the minimal achievable (subject to specifications (i)-(iii)) settling time  $T_*$ ;
- (c) Explain how linear programming and Q-parameterization can be used to find finite order controllers which achieve near-optimal (subject to specifications (i)-(iii)) settling times;
- (d) Implement the algorithms from (a) and (b) in MATLAB to find a controller with a guaranteed level of optimality (subject to specifications (i)-(iii)) for the settling time.

**Problem 9.3**

It is known that, for a stable LTI system

$$G = [G_1 \ G_2 \ \dots \ G_m]$$

with  $m$  scalar inputs  $w_1, w_2, \dots, w_m$  and one output  $e$ , the maximal energy  $E = E(G)$  of  $e$  achievable (subject to zero initial conditions) when the energy of each of the signals  $w_k$  is bounded by one, equals the supremum of the square of the H-Infinity norm of

$$G_\tau = [\tau_1 G_1 \ \tau_2 G_2 \ \dots \ \tau_m G_m],$$

where  $\tau_k > 0$  are positive numbers satisfying

$$\sum_{k=1}^m \frac{1}{\tau_k^2} = 1.$$

- (a) Explain how the KYP lemma can be used to reduce the task of finding  $E = E(G)$ , where  $G$  is given by its state space model

$$e = Cx + Dw, \quad \dot{x} = Ax + Bw, \quad w = \begin{bmatrix} w_1 \\ \vdots \\ w_m \end{bmatrix},$$

to semidefinite programming;

- (b) Implement the algorithm from (a) in MATLAB, and test it on

$$G(s) = \frac{[s \quad s-1]}{s^2 + s + 1}.$$