6.055J / 2.038J The Art of Approximation in Science and Engineering
Spring 2008

6.055J/2.038J (Spring 2008)

Solution set 4

Do the following warmups and problems. Due in class on ***Friday, 04 Apr 2008****.*

Open universe: *Collaboration, notes, and other sources of information are* ***encouraged.*** *However, avoid looking up answers until you solve the problem (or have tried hard). That policy helps you learn the most from the problems.*

Bring a photocopy to class on the due date*, trade it for a solution set, and figure out or ask me about any confusing points. Your work will be graded lightly: P (made a reasonable effort), D (did not make a reasonable effort), or F (did not turn in).*

Warmups

1. Minimum power

In lecture we estimated the flight speed that minimizes energy consumption. Call that speed v_E. We could also have estimated v_P, the speed that minimizes power consumption. What is the ratio v_P/v_E?

The zillions of constants (such as ρ) clutter the analysis without changing the result. So I'll simplify the problem by using a system of units where all the constants are 1. Then the energy is

$$E \sim v^2 + \frac{1}{v^2},$$

where the first term is from drag and the second term is from lift. The power is energy per time, and time is inversely proportional to v, so $P \propto Ev$ and

$$P \sim v^3 + \frac{1}{v}.$$

The first term is the steep v^3 dependence of drag power on velocity (which we used to estimate the world-record cycling and swimming speeds).

The energy expression is unchanged when $v \to 1/v$, so it has a minimum at $v_E = 1$. To minimize the power, use calculus (ask me if you are curious about calculus-free ways to minimize it):

$$\frac{dP}{dv} \sim 3v^2 - \frac{1}{v^2} = 0,$$

therefore $v_P = 3^{-1/4}$ (roughly $3/4$), which is also the ratio v_P/v_E.

So the minimum-power speed is about 25% less than the minimum-energy speed. That result makes sense. Drag power grows very fast as v increases – much faster than lift power decreases – so it's worth reducing the speed a little to reduce the drag a lot.

If you don't believe the simplification that I used of setting all constants to 1 – and it is not immediately obvious that it should work – then try using this general form:

$$E \sim Av^2 + \frac{B}{v^2},$$

where A and B are constants. You'll find that v_E and v_P get the same function of A and B, which disappears from the ratio v_P/v_E.

2. Solitaire

You start with the numbers 3, 4, and 5. At each move, you choose any two of the three numbers – call the choices a and b – and replace them with $0.8a - 0.6b$ and $0.6a + 0.8b$. The goal is to reach 4, 4, 4. Can you do it? If yes, give a move sequence; if no, show that you cannot.

To see whether solitaire games are solvable, look for an invariant. Alas there is no algorithm for finding invariants; you have to use clues and make lucky guesses.

Speaking of clues, is it a happy coincidence that $0.8^2 + 0.6^2 = 1$? That convenient sum suggests looking at sums of squares, and how those are changed by making a move. Replacing a and b by $a' = 0.8a - 0.6b$ and $b' = 0.6a + 0.8b$ makes the sum of squares $a^2 + b^2$ into $a'^2 + b'^2$. Expand that expression:

$$\begin{aligned} a'^2 + b'^2 &= (0.8a - 0.6b)^2 + (0.6a + 0.8b)^2 \\ &= 0.64a^2 - 0.96ab + 0.36b^2 + 0.36a^2 + 0.96ab + 0.64b^2 \\ &= a^2 + b^2. \end{aligned}$$

Great! Each move leaves the sum of squares unchanged. That sum started out with the invariant at $3^2 + 4^2 + 5^2 = 50$, so it remains 50. The goal state, however, requires that the invariant become $4^2 + 4^2 + 4^2 = 48$. It's not possible to reach the goal.

The invariant has a nice geometric interpretation (a picture). To see it, let $P = (a, b, c)$ be the coordinates of a point in three-dimensional space. Then each move leaves unchanged the distance to the origin, which is $\sqrt{a^2 + b^2 + c^2}$. So each move shifts P to another location equally distant from the origin, meaning that it moves P on the surface of a sphere. But it cannot escape the surface.

An interesting question to which I don't know the answer: Can you reach every point on the surface of the sphere? The distance invariant does not forbid it, but maybe other constraints do?

Problems

3. Bird flight

a. For geometrically similar animals (same shape and composition but different size), how does the minimum-energy speed v depend on mass M and air density ρ? In other words, what are the exponents α and β in $v \propto \rho^\alpha M^\beta$?

From the lecture notes,

$$Mg \sim C^{1/2}\rho v^2 L^2,$$

where C is the modified drag coefficient. So

$$v \sim \left(\frac{Mg}{C^{1/2}\rho L^2}\right)^{1/2}.$$

For geometrically similar animals, g is independent of size (they all fight the same gravity) and C is also independent of size (because the drag coefficient depends only on shape). But M depends on L according to $M \propto L^3$ or $L \propto M^{1/3}$. So the L^2 in the denominator is proportional to $M^{2/3}$ making

$$v \propto \rho^{-1/2} M^{1/6}.$$

giving $\alpha = -1/2$ and $\beta = 1/6$.

The inverse relationship between the speed and density explains why planes fly at a high altitude. The energy consumption at the minimum-energy speed is independent of ρ, so by flying high where ρ is low, planes increase their speed without increasing their energy consumption.

b. Use that result to write the ratio $v_{747}/v_{\text{godwit}}$ as a product of dimensionless factors, where v_{747} is the minimum-energy speed of a 747, and v_{godwit} is the minimum-energy speed of a bar-tailed godwit. Then estimate the dimensionless factors and their product. Useful information: $m_{\text{godwit}} \sim 0.4\,\text{kg}$.

Assuming that the animals and planes fly at the minimum-energy speed,

$$\frac{v_{747}}{v_{\text{godwit}}} = \left(\frac{\rho_{\text{high}}}{\rho_{\text{sealevel}}}\right)^{-1/2} \cdot \left(\frac{m_{747}}{m_{\text{godwit}}}\right)^{1/6}.$$

A plane flies at around 10 km where the density is roughly one-third of the sea-level density. The mass of a 747 is roughly $4 \cdot 10^5$ kg, so the mass ratio is 10^6. Therefore the speed ratio should be roughly

$$(1/3)^{-1/2} \times (10^6)^{1/6} = \sqrt{3} \times 10 \sim 17.$$

c. Use v_{747}, from experience or from looking it up, to find v_{godwit}. Compare with the speed of the record-setting bar-tailed godwit, which made its 11,570 km journey in 8 days, 12 hours.

A 747 flies at around 600 mph so the godwit should fly around 600/17 mph $\sim$ 35 mph. The speed of record-setting godwit is

$$\frac{11,570\,\text{km}}{8.5\,\text{days}} \times \frac{0.6\,\text{mi}}{1\,\text{km}} \times \frac{1\,\text{day}}{24\,\text{hours}} \sim 35\,\text{mph}.$$

That's absurdly close.

4. Hovering and hummingbirds

A simple model of hovering is that the animal or helicopter (mass M and wingspan L) forces air downward to stay aloft.

a. Estimate the downward air speed v_{down} needed to hover.

The reasoning is the same as for a forward-flying animal: It must deflect air downwards in order to recoil upwards and stay aloft. The hummingbird sweeps air downwards roughly over an area L^2, so in a time t, it has swept a volume $L^2 v_{\text{down}} t$ and a mass

$$m_{\text{air}} \sim \rho L^2 v_{\text{down}} t.$$

To get the right recoil, the momentum provided by gravity must be the momentum imparted to the air. Gravity provides a force Mg, so in time t it provides a momentum Mgt (since $F = d(\text{momentum})/dt$). So $m_{\text{air}} v_{\text{down}} \sim Mgt$ and

$$v_{\text{down}} \sim \frac{Mgt}{m_{\text{air}}} \sim \frac{Mgt}{\rho L^2 v_{\text{down}} t}.$$

Before solving for v_{down}, note the difference between this analysis and the analysis for forward-flying animals. In that analysis (the one given in the notes), the similar formula has v_{forward} in the denominator on the right side rather than v_{down}.

Solving for v_{down} gives

$$v_{\text{down}} \sim \left(\frac{Mg}{\rho L^2}\right)^{1/2}.$$

b. Show that the power required to hover is

$$P \sim \frac{(Mg)^{3/2}}{\rho^{1/2} L}.$$

Power is force times velocity. The bird is generating a lift force Mg so that it can hover, and forcing air downward at speed v_{down}, so the power is

$$P \sim Mg v_{\text{down}} \sim Mg \left(\frac{Mg}{\rho L^2}\right)^{1/2} = \frac{(Mg)^{3/2}}{\rho^{1/2} L}.$$

c. Estimate P and v_{down} for a person hovering by flapping or waving his or her arms.

I estimate v_{down} using a wingspan of $L \sim 2\,\text{m}$ and a mass of $M \sim 70\,\text{kg}$:

$$v_{\text{down}} \sim \left(\frac{70\,\text{kg} \times 10\,\text{m}\,\text{s}^{-2}}{1\,\text{kg}\,\text{m}^{-3} \times 4\,\text{m}^2}\right)^{1/2} \sim 14\,\text{m}\,\text{s}^{-1}.$$

This value is probably an underestimate because it assumes that the person's arm motions fill the whole area L^2, whereas it might fill only one-fifth or even one-tenth of the whole area.

But leaving the estimate as is, the resulting power is

$$P \sim Mg v_{\text{down}} \sim 70\,\text{kg} \times 10\,\text{m}\,\text{s}^{-2} \times 14\,\text{m}\,\text{s}^{-1} \sim 10^4\,\text{W}.$$

That power is significantly greater than the 500 W endurance-power that we estimated for an athlete. People have no chance of hovering, at least not without using very large (and very light!) wings.

d. How does P depend on M for geometrically similar animals (same composition and shape but varying size)? In other words, give the exponent β in

$$P \propto M^{\beta}.$$

Again $L \propto M^{1/3}$, so

$$P \sim \frac{(Mg)^{3/2}}{\rho^{1/2}L}$$

becomes

$$P \propto \frac{M^{3/2}}{M^{1/3}} = M^{5/6}.$$

So the power increases slightly more slowly than the mass does.

e. What fraction of its body weight does a hummingbird ($M \sim 3\,\mathrm{g}$) eat every day in order to hover for a working day (8 hours)? Compare to the fraction for a person in a typical day. [Hummingbirds eat nectar, which is roughly equal parts sugar and water.]

For simplicity I assume that humans and hummingbirds are geometrically similar, so $L \propto M^{1/3}$; then I use that proportionality to work out how the body-weight fraction scales with M.

The body-weight fraction is proportional to the hovering power per body mass P/M, which is proportional to $M^{-1/6}$. So small animals need to eat a larger fraction of their body mass to hover than do large animals. The mass ratio between a hummingbird and a person is roughly $2.5 \cdot 10^4$, or $10^6/40$, so ratio of body-weight fractions is $(10^6/40)^{-1/6}$ or $40^{1/6}/10$. Since $40^{1/6}$ is roughly 2, the ratio is roughly 0.2.

Now I compute the body-weight fraction for a hovering person starting from the power required to hover. In part (c) I computed that a hovering person requires roughly $10^4\,\mathrm{W}$. Pretend that humans also eat nectar. Sugar provides $4\,\mathrm{kcal/\,g}$ of energy, but the conversion is about 25% efficient, so about $1\,\mathrm{kcal/\,g}$ of mechanical power. That's $4\,\mathrm{MJ\,kg^{-1}}$. Nectar is one-half as useful, so it provides $2\,\mathrm{MJ\,kg^{-1}}$.

One day of hovering (8 hours) means eating (or drinking) this much nectar:

$$10^4\,\mathrm{W} \times 8\,\mathrm{hours} \times \frac{3600\,\mathrm{s}}{1\,\mathrm{hour}} \times \frac{1\,\mathrm{kg\ nectar}}{2 \cdot 10^6\,\mathrm{J}} \sim 150\,\mathrm{kg}.$$

So a hovering person would eat roughly double their body weight in nectar, and a hummingbird (using the estimated ratio of 0.2) would eat $2 \times 0.2 \sim 50\%$ of their body weight in nectar!

Hummingbirds are hungry animals! And big hummingbirds would have a hard time hovering because the energy requirements are so large.

In a typical human day – definitely not spent hovering – we eat 2500 kcal or roughly 500 g of carbohydrate. Most foods (e.g. plants, meat) are about 80 or 90% water, so we probably eat 5 or 10 times 500 g in mass. I'll use a factor of 7 because the numbers turn out simple. A factor of 7 makes the total mass consumed in a day roughly 3.5 kg; that mass is 0.05 times a typical human mass.

Optional

5. Inertia tensor

[For those who know about inertia tensors.] Here is the inertia tensor (the generalization of moment of inertia) of a particular object, calculated in a lousy coordinate system:

$$\begin{pmatrix} 4 & 0 & 0 \\ 0 & 5 & 4 \\ 0 & 4 & 5 \end{pmatrix}$$

a. Change coordinate systems to a set of principal axes. In other words, write the inertia tensor as

$$\begin{pmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{pmatrix}$$

and give I_{xx}, I_{yy}, and I_{zz}. *Hint:* What properties of a matrix are invariant when changing coordinate systems?

Whatever coordinate change I make, I will leave the x axis alone because the I_{xx} component is already separated from the y- and z submatrix. That submatrix is

$$\begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}$$

I have to figure out how changing the coordinate system changes this submatrix. Rather than find the coordinate change explicitly, I use invariants to avoid that computation.

One invariant of any matrix, not just of this 2×2 matrix, is its determinant. Another invariant is its trace (the sum of the diagonal elements). In the nasty coordinate system, the trace of the y- and z submatrix is $5 + 5 = 10$. So the trace is 10 in the nice coordinate system. The determinant is $5 \times 5 - 4 \times 4 = 9$, so it the determinant is 9 in the nice coordinate system.

Those facts are sufficient to deduce the submatrix in the nice coordinate system (without needing to figure out what the nice coordinate system is). In the nice coordinate system, the 2×2 submatrix looks like

$$\begin{pmatrix} I_{yy} & 0 \\ 0 & I_{zz} \end{pmatrix}$$

So I need to find I_{yy} and I_{zz} such that

$$I_{yy} + I_{zz} = 10 \quad \text{(from the trace invariant)}$$

and

$$I_{yy} I_{zz} = 9 \quad \text{(from the determinant invariant)}$$

The solution is $I_{yy} = 1$ and $I_{zz} = 9$ (or vice versa). So the inertia tensor becomes

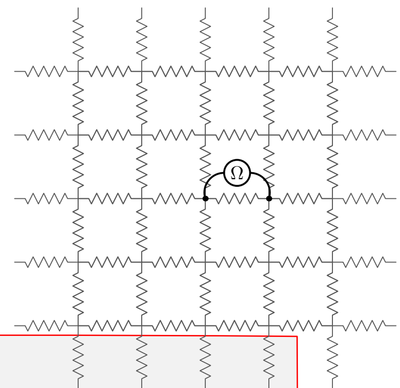
$$\begin{pmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 9 \end{pmatrix}$$

b. Give an example of an object with a similar inertia tensor. On Friday in class we'll have a demonstration.

The object has three principal axes, each with a different moment of inertia. If the object is rectangular and uniform density, the three axes must have different lengths. Most books fit into this category. They have a short axis that passes perpendicularly through the pages (this axis is the one with the highest moment of inertia). The medium-length axis is perpendicular to the spine. And the long axis is parallel to the spine.

6. Resistive grid

In an infinite grid of 1-ohm resistors, what is the resistance measured across one resistor?



To measure resistance, an ohmmeter injects a current I at one terminal (for simplicity, say $I = 1$ A), removes the same current from the other terminal, and measures the resulting voltage difference V between the terminals. The resistance is $R = V/I$.

Hint: Use symmetry. But it's still a hard problem!

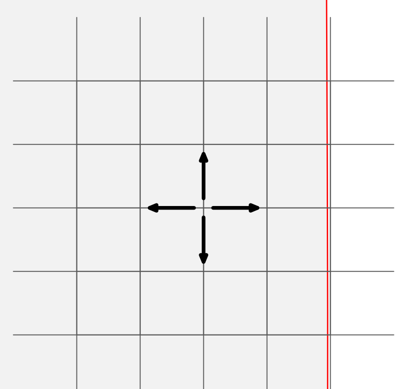
I'd like to find the current flowing through the resistor when 1 A is sent into one terminal of the ohmmeter and removed from its other terminal. The solution has two steps, each subtle:

1. Break the resistance-measuring experiment into two parts, each having a lot of symmetry.

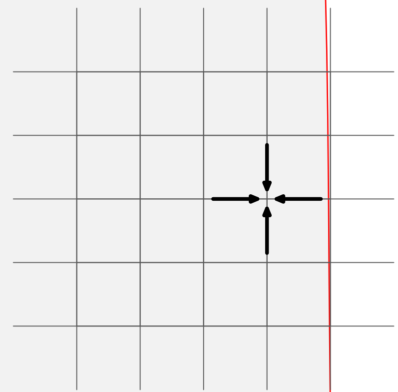
2. Analyze those parts using symmetry.

The current distribution that results from the full resistance-measuring experiment is not sufficiently symmetric because it has a preferred direction along the selected resistor. However, if I break the experiment into two parts – inserting current and removing current – then each part produces a symmetric current distribution.

By symmetry – because all four coordinate directions are equivalent – inserting 1 A produces 1/4 A flowing in each coordinate direction away from the terminal. Let's call this terminal the positive terminal. So inserting the 1 A at the positive terminal produces 1/4 A through the selected resistor, and this current flows away from the positive terminal.



By symmetry, removing 1 A produces 1/4 A in each coordinate direction, flowing toward the terminal. Let's call this terminal the negative terminal. So removing 1 A produces 1/4 A through the selected resistor, flowing toward the negative terminal. Equivalently, it produces 1/4 A flowing away from the positive terminal.



Now superimpose the two pictures to reproduce the experiment of measuring the resistance. The experiment produces 1/2 A through the resistor, flowing from the positive to the negative terminal. The voltage across the resistor is the current times its resistance, so the voltage is 1/2 V. Since a 1 A test current produces a 1/2 V drop, the effective resistance is 1/2 Ω.

If you want an even more difficult problem: Find the resistance measured across a diagonal!