## In-Class Problems Week 10, Mon.

## Problem 1.

Recall that for functions $f, g$ on $\mathbb{N}, f=O(g)$ iff

$$
\begin{equation*}
\exists c \in \mathbb{N} \exists n_{0} \in \mathbb{N} \forall n \geq n_{0} \quad c \cdot g(n) \geq|f(n)| \tag{1}
\end{equation*}
$$

For each pair of functions below, determine whether $f=O(g)$ and whether $g=O(f)$. In cases where one function is O() of the other, indicate the smallest nonnegative integer, $c$, and for that smallest $c$, the smallest corresponding nonnegative integer $n_{0}$ ensuring that condition (1) applies.
(a) $f(n)=n^{2}, g(n)=3 n$.
(b) $f(n)=(3 n-7) /(n+4), g(n)=4$
(c) $f(n)=1+(n \sin (n \pi / 2))^{2}, g(n)=3 n$

## Problem 2.

(a) Indicate which of the following asymptotic relations below on the set of nonnegative real-valued functions are equivalence relations $(\mathbf{E})$, strict partial orders $(\mathbf{S})$, weak partial orders $(\mathbf{W})$, or none of the above ( $\mathbf{N}$ ).

- $f \sim g$, the "asymptotically equal" relation.
- $f=o(g)$, the "little Oh" relation.
- $f=O(g)$, the "big Oh" relation.
- $f=\Theta(g)$, the "Theta" relation.
- $f=O(g)$ AND NOT $(g=O(f))$.
(b) Indicate the implications among the assertions in part (a). For example,

$$
f=o(g) \text { IMPLIES } f=O(g)
$$

## Problem 3.

## False Claim.

$$
\begin{equation*}
2^{n}=O(1) \tag{2}
\end{equation*}
$$

Explain why the claim is false. Then identify and explain the mistake in the following bogus proof.

[^0]Bogus proof. The proof is by induction on $n$ where the induction hypothesis, $P(n)$, is the assertion (2). base case: $P(0)$ holds trivially.
inductive step: We may assume $P(n)$, so there is a constant $c>0$ such that $2^{n} \leq c \cdot 1$. Therefore,

$$
2^{n+1}=2 \cdot 2^{n} \leq(2 c) \cdot 1
$$

which implies that $2^{n+1}=O(1)$. That is, $P(n+1)$ holds, which completes the proof of the inductive step.
We conclude by induction that $2^{n}=O(1)$ for all $n$. That is, the exponential function is bounded by a constant.

## Supplemental problems

## Problem 4.

Assign true or false for each statement and prove it.

- $n^{2} \sim n^{2}+n$
- $3^{n}=O\left(2^{n}\right)$
- $n^{\sin (n \pi / 2)+1}=o\left(n^{2}\right)$
- $n=\Theta\left(\frac{3 n^{3}}{(n+1)(n-1)}\right)$


## Problem 5.

Give an elementary proof (without appealing to Stirling's formula) that $\log (n!)=\Theta(n \log n)$.

MIT OpenCourseWare
https://ocw.mit.edu

### 6.042J / 18.062J Mathematics for Computer Science

 Spring 2015For information about citing these materials or our Terms of Use, visit: https://ocw.mit.edu/terms.


[^0]:    2015, Eric Lehman, F Tom Leighton, Albert R Meyer. This work is available under the terms of the Creative Commons Attribution-NonCommercial-ShareAlike 3.0 license.

