Quiz 2

Your name:____

Circle the name of your Tutorial Instructor:

David Hanson Jelani Sayan

- This quiz is **closed book** except for a personal one-page crib sheet. No calculators either.
- There are 6 problems totaling 100 points. Problems are labeled with their point values.

  You may assume any of the results presented in class or in the lecture notes. Total time is 80 minutes.
- Write your solutions in the space provided. If you need more space, write on the back of the sheet containing the problem. Please keep your entire answer to a problem on that problem's page. Incorrect short answers can receive partial credit only if work is clearly shown.
- GOOD LUCK!

DO NOT WRITE BELOW THIS LINE

Problem	Points	Grade	Grader
1	10		
2	20		
3	15		
4	20		
5	10		
6	25		
Total	100		

Problem 1 (10 points). True/False. Circle the appropriate answer.

(a) (2 points) A state machine with a strictly decreasing derived variable must terminate.

True **False**

(b) (2 points) In a set of stable marriages with more than one couple, it is possible for everyone to be married to the person at the bottom of their preference list.

True **False**

(c) (2 points) An infinite geometric sum whose ratio between successive terms is r converges if $|r| \leq 1$.

True **False**

(d) (2 points) Four books can be stacked at the edge of a table so that the top book lies completely over the edge of the table.

True **False**

(e) (2 points) The security of RSA relies on the assumption that the ability to decipher RSA-encrypted messages efficiently would imply the ability to factor key-sized numbers efficiently.

True **False**

Problem 2 (20 points). **(a) (10 points)** Using the Pulverizer, find an x in the range $[0..99]$ such that x is an inverse of 19 modulo 100.

(b) (5 points) What is the value of $\phi(360)$, where ϕ is Euler's function? ________

(c) (5 points) What is the value of $7^{98} \text{ rem } 360$? ________

Problem 3 (15 points). Consider the following functions:

$$\begin{array}{llll} f_1(n) = \log(n!) & f_2(n) = 10^{100} & f_3(n) = \sum_{i=1}^{n}(1/i) & f_4(n) = \log(4^n) \\ f_5(n) = \log(n^n) & f_6(n) = 2 + \sin n & f_7(n) = \log(6^n) & \end{array}$$

(a) (10 points) List the digits $1, \ldots, 7$ in an order such that if digit i comes before j in your list, then $f_i = O(f_j)$.

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(b) (5 points) List a sequence of sets of the digits so that i and j are in the same set iff $f_i = \Theta(f_j)$. Write your list in a form such as "$\{543\}, \{76\}, \{21\}$".

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Problem 4 (20 points). Write simple formulas for the following quantities. You do not have to calculate numerical values for the formulas.

(a) (6 points) The number of rearrangements of the word $BAZOOKA$ in which the two O's do not appear next to each other.

(b) (6 points) The number of rearrangements of the word $BAZOOKA$ in which the two O's do not appear next to each other *and* that do not start with B.

(c) (2 points) The number of *nonnegative integer* solutions to the equality:

$$x_1 + x_2 + \ldots + x_{10} = 100.$$

(d) (6 points) The number of *positive integer* solutions to the inequality:

$$x_1 + x_2 + \ldots + x_{10} \leq 100.$$

Problem 5 (10 points). To prove that $n^4 = O(\sum_{i=1}^{n} i^3)$ we can use the integral method to bound the sum. In particular, we should obtain a(n)

upper lower (CIRCLE THE RIGHT CHOICE)

bound on the sum that is equal to the value of $\int_a^b (x+c)^d dx$ where

a is ______, b is ______, c is ______, and d is ______.

Problem 6 (25 points). We will describe a process that operates on sequences of numbers. The process will start with a sequence that is some *permutation* of the length $4n$ sequence

$$(1, 2, \ldots, n, 1, 2, \ldots, n, 1, 2, \ldots, 2n).$$

(a) (7 points) Write a simple formula for the number of possible starting sequences.

If $(s_1, \ldots, s_k)$ is a sequence of numbers, then the i and jth elements of the sequence are *out of order* if the number on the left is strictly larger the number on the right, that is, if $i < j$ and $s_i > s_j$. Otherwise, the ith and jth elements are *in order*. Define $p(S)$::= the number of "out-of-order" pairs of elements in a sequence, S.

For example, if S is the sequence

$$(3, 4, 2, 1, 7, 3),$$

then the 1st and 3rd elements of S, (namely, 3 and 2), are out of order, but the 3rd and 6th elements (2 and 3) are in order. The 1st and 6th elements of S are also in order, since they are both 3. In this case, $p(S) = 7$.

The *reversal* of $(s_1, \ldots, s_k)$ is $(s_k, \ldots, s_1)$. So the reversal of the sequence S is

$$(3, 7, 1, 2, 4, 3).$$

From the starting sequence, we carry out the following process:

(*) Pick two consecutive elements in the current sequence, say the ith and $(i + 1)$st.

I. If the elements are not in order, then **switch them** in the sequence and repeat step (*).

II. If the elements are in order, **remove both**, resulting in a sequence that is shorter by two. If the length of the resulting sequence is zero or one, the process is over. Otherwise, **reverse the sequence** and repeat step (*).

This process can be modelled as a state machine where the states are the sequences that appear at step (*). We then consider the derived variables on the following page:

i. $p(S)$ _____

ii. $\text{length}(S) \text{ rem } 2$ _____

iii. $\text{length}(S) + p(S)$ _____

iv. $\max(p(S), 8n^2)$ _____

v. $4n^2 \cdot \text{length}(S) + p(S)$ _____

(b) (14 points) Indicate next to each of the derived variables above which one of these properties it has:

constant	C
strictly increasing	SI
strictly decreasing	SD
weakly increasing but not constant	WI
weakly decreasing but not constant	WD
none of the above	N

(c) (2 points) Which of the variable behaviors in i.–v. above immediately implies that the process will definitely terminate? ______

(d) (2 points) Which of the variable behaviors in i.–v. above immediately implies that, starting from any of the possible starting states from part (a), the process will not terminate with a length 1 sequence? ______