CHAPTER FOUR

some basic probabilistic processes

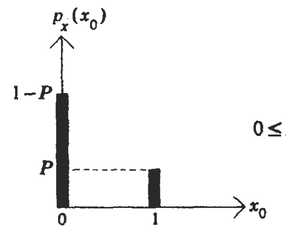
This chapter presents a few simple probabilistic processes and develops family relationships among the PMF's and PDF's associated with these processes.

Although we shall encounter many of the most common PMF's and PDF's here, it is not our purpose to develop a general catalogue. A listing of the most frequently occurring PMF's and PDF's and some of their properties appears as an appendix at the end of this book.

4-1 The Bernoulli Process

A single *Bernoulli trial* generates an experimental value of discrete random variable x, described by the PMF

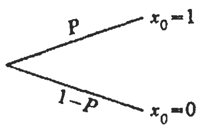
$$p_x(x_0) = \begin{cases} 1-P & x_0 = 0 \\ P & x_0 = 1 \\ 0 & \text{otherwise} \end{cases} \qquad 0 \le P \le 1$$



Random variable x, as described above, is known as a *Bernoulli random variable*, and we note that its PMF has the z transform

$$p_x^T(z) = \sum_{x_0} z^{x_0} p_x(x_0) = z^0(1-P) + zP = 1 - P + zP$$

The sample space for each Bernoulli trial is of the form



Either by use of the transform or by direct calculation we find

$$E(x) = P \qquad E(x^2) = P \qquad \sigma_x^2 = P(1-P)$$

We refer to the outcome of a Bernoulli trial as a *success* when the experimental value of x is unity and as a *failure* when the experimental value of x is zero.

A *Bernoulli process* is a series of independent Bernoulli trials, each with the same probability of success. Suppose that n independent Bernoulli trials are to be performed, and define discrete random variable k to be the number of successes in the n trials. Random variable k is noted to be the sum of n independent Bernoulli random variables, so we must have

$$p_k^T(z) = [p_x^T(z)]^n = (1 - P + zP)^n$$

There are several ways to determine $p_k(k_0)$, the probability of exactly k_0 successes in n independent Bernoulli trials. One way would be to apply the binomial theorem

$$(a+b)^n = \sum_{l=0}^{n} \binom{n}{l} a^l b^{n-l}$$

to expand $p_k^T(z)$ in a power series and then note the coefficient of z^{k_0} in this expansion, recalling that any z transform may be written in the form

$$p_k^T(z) = p_k(0) + zp_k(1) + z^2p_k(2) + \cdots$$

This leads to the result known as the *binomial* PMF,

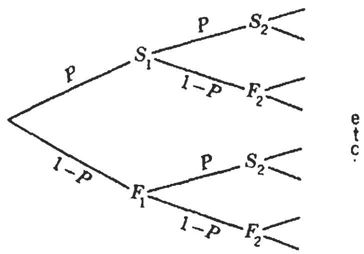
$$p_k(k_0) = \binom{n}{k_0} P^{k_0}(1-P)^{n-k_0} \qquad k_0 = 0, 1, 2, \ldots, n$$

where the notation is the common

$$\binom{n}{k_0} = \frac{n!}{(n-k_0)!k_0!}$$

discussed in Sec. 1-9.

Another way to derive the binomial PMF would be to work in a sequential sample space for an experiment which consists of n independent Bernoulli trials,



We have used the notation

$$\begin{Bmatrix} S_n \\ F_n \end{Bmatrix} = \begin{Bmatrix} \text{success} \\ \text{failure} \end{Bmatrix} \text{ on the } n\text{th trial}$$

Each sample point which represents an outcome of exactly k_0 successes in the n trials would have a probability assignment equal to $P^{k_0}(1-P)^{n-k_0}$. For each value of k_0, we use the techniques of Sec. 1-9 to determine that there are $\binom{n}{k_0}$ such sample points. Thus, we again obtain

$$p_k(k_0) = \binom{n}{k_0} P^{k_0}(1-P)^{n-k_0} \qquad k_0 = 0, 1, 2, \ldots, n$$

for the binomial PMF.

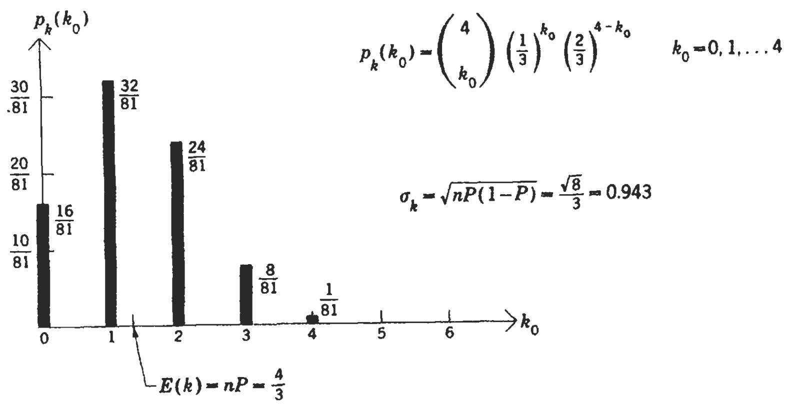
We can determine the expected value and variance of the *binomial random variable* k by any of three techniques. (One should always review his arsenal before selecting a weapon.) To evaluate $E(k)$ and σ_k^2 we may

1 Perform the expected value summations directly.
2 Use the moment-generating properties of the z transform, introduced in Sec. 3-3.
3 Recall that the expected value of a sum of random variables is *always* equal to the sum of their expected values and that the variance of a sum of *linearly independent* random variables is equal to the sum of their individual variances.

Since we know that binomial random variable k is the sum of n independent Bernoulli random variables, the last of the above methods is the easiest and we obtain

$$E(k) = nE(x) = nP \qquad \sigma_k^2 = n\sigma_x^2 = nP(1 - P)$$

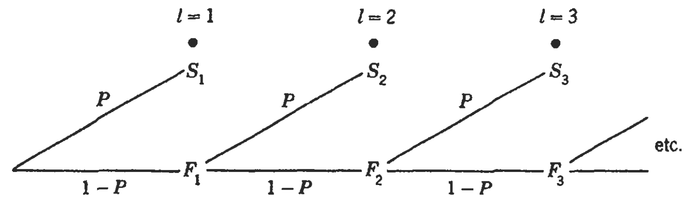
Before moving on to other aspects of the Bernoulli process, let's look at a plot of a binomial PMF. The following plot presents $p_k(k_0)$ for a Bernoulli process, with $P = \frac{1}{3}$ and $n = 4$.



4-2 Interarrival Times for the Bernoulli Process

It is often convenient to refer to the successes in a Bernoulli process as *arrivals*. Let discrete random variable l_1 be the number of Bernoulli trials up to and including the first success. Random variable l_1 is known as the *first-order interarrival time*, and it can take on the experimental values 1, 2, ... We begin by determining the PMF $p_{l_1}(l)$. (Note that since we are subscripting the random variable there is no reason to use a subscripted dummy variable in the argument of the PMF.)

We shall determine $p_{l_1}(l)$ from a sequential sample space for the experiment of performing independent Bernoulli trials until we obtain our first success. Using the notation of the last section, we have



We have labeled each sample point with the experimental value of random variable l_1 associated with the experimental outcome represented by that point. From the above probability tree we find that

$$p_{l_1}(l) = P(1 - P)^{l-1} \qquad l = 1, 2, \ldots$$

and since its successive terms decrease in a geometric progression, this PMF for the first-order interarrival times is known as the *geometric* PMF. The z transform for the geometric PMF is

$$p_{l_1}{}^T(z) = \sum_{l=0}^{\infty} p_{l_1}(l)z^l = \sum_{l=1}^{\infty} P(1 - P)^{l-1}z^l = \frac{zP}{1 - z(1 - P)}$$

Since direct calculation of $E(l_1)$ and $\sigma_{l_1}{}^2$ in an l_1 event space involves difficult summations, we shall use the moment-generating property of the z transform to evaluate these quantities.

$$E(l_1) = \left[\frac{d}{dz} p_{l_1}{}^T(z)\right]_{z=1} = \frac{1}{P}$$

$$\sigma_{l_1}{}^2 = \left\{\frac{d^2}{dz^2} p_{l_1}{}^T(z) + \frac{d}{dz} p_{l_1}{}^T(z) - \left[\frac{d}{dz} p_{l_1}{}^T(z)\right]^2\right\}_{z=1} = \frac{1 - P}{P^2}$$

Suppose that we were interested in the conditional PMF for the *remaining number of trials* up to and including the next success, given that there were no successes in the first m trials. By conditioning the event space for l_1 we would find that the conditional PMF for $y = l_1 - m$, the remaining number of trials until the next success, is still a geometric random variable with parameter P (see Prob. 4.03). This is a result attributable to the "no-memory" property (independence of trials) of the Bernoulli process. The PMF $p_{l_1}(l)$ was obtained as the PMF for the number of trials up to and including the first success. Random variable l_1, the first-order interarrival time, represents both the waiting time (number of trials) from one success through the next success and the waiting time from *any* starting time through the next success.

Finally, we wish to consider the higher-order interarrival times for a Bernoulli process. Let random variable l_r, called the *rth-order interarrival time*, be the number of trials up to and including the rth success. Note that l_r is the sum of r independent experimental values of random variable l_1; so we must have

$$p_{l_r}{}^T(z) = [p_{l_1}{}^T(z)]^r = \left[\frac{zP}{1 - z(1-P)}\right]^r$$

There are several ways we might attempt to take the inverse of this transform to obtain $p_{l_r}(l)$, the PMF for the rth-order interarrival time, but the following argument seems both more intuitive and more efficient. Since $p_{l_r}(l)$ represents the probability that the rth success in a Bernoulli process arrives on the lth trial, $p_{l_r}(l)$ may be expressed as

$$p_{l_r}(l) = \begin{pmatrix}\text{probability of having}\\ \text{exactly } r-1 \text{ successes}\\ \text{in the first } l-1 \text{ trials}\end{pmatrix} \times \begin{pmatrix}\text{conditional probability of hav-}\\ \text{ing } r\text{th success on the } l\text{th trial,}\\ \text{given exactly } r-1 \text{ successes}\\ \text{in the previous } l-1 \text{ trials}\end{pmatrix}$$

The first term in the above product is the binomial PMF evaluated for the probability of exactly $r - 1$ successes in $l - 1$ trials. Since the outcome of each trial is independent of the outcomes of all other trials, the second term in the above product is simply equal to P, the probability of success on any trial. We may now substitute for all the words in the above equation to determine the PMF for the rth-order interarrival time (the number of trials up to and including the rth success) for a Bernoulli process

$$p_{l_r}(l) = \left[\binom{l-1}{r-1} P^{r-1}(1-P)^{l-1-(r-1)}\right] P$$

$$= \binom{l-1}{r-1} P^r(1-P)^{l-r} \qquad l = r, r+1, r+2, \ldots; \quad r = 1, 2, 3, \ldots$$

Of course, with $r = 1$, this yields the geometric PMF for l_1 and thus provides an alternative derivation of the PMF for the first-order interarrival times. The PMF $p_{l_r}(l)$ for the number of trials up to and including the rth success in a Bernoulli process is known as the *Pascal* PMF. Since l_r is the sum of r independent experimental values of l_1, we have

$$E(l_r) = rE(l_1) = \frac{r}{P} \qquad \sigma_{l_r}{}^2 = r\sigma_{l_1}{}^2 = \frac{r(1-P)}{P^2}$$

The *negative binomial PMF*, a PMF which is very closely related to the Pascal PMF, is noted in Prob. 4.01.

As one last note regarding the Bernoulli process, we recognize that the relation

$$\sum_{r=1}^{\infty} p_{l_r}(l) = P \qquad l = 1, 2, \ldots$$

is always true. The quantity $p_{l_r}(l)$, evaluated for any value of l, equals the probability that the rth success will occur on the lth trial. Any success on the lth trial must be the first, or second, or third, etc., success after $l = 0$. Therefore, the sum of $p_{l_r}(l)$ over r simply represents the probability of a success on the lth trial. From the definition of the Bernoulli process, this probability is equal to P.

Our results for the Bernoulli process are summarized in the following section.

4-3 Summary of the Bernoulli Process

Each performance of a Bernoulli trial generates an experimental value of the *Bernoulli* random variable x described by

$$p_x(x_0) = \begin{cases} 1-P & x_0 = 0 \text{ (a "failure")} \\ P & x_0 = 1 \text{ (a "success")} \end{cases}$$

$$p_x{}^T(z) = 1 - P + zP \qquad E(x) = P \qquad \sigma_x{}^2 = P(1-P)$$

A series of identical independent Bernoulli trials is known as a *Bernoulli process.* The number of successes in n trials, random variable k, is the sum of n independent Bernoulli random variables and is described by the *binomial* PMF

$$p_k(k_0) = \binom{n}{k_0} P^{k_0}(1-P)^{n-k_0} \qquad k_0 = 0, 1, 2, \ldots, n$$

$$p_k{}^T(z) = (1 - P + zP)^n \qquad E(k) = nP \qquad \sigma_k{}^2 = nP(1-P)$$

The number of trials up to and including the first success is described by the PMF for random variable l_1, called the *first-order interarrival (or waiting) time.* Random variable l_1 has a *geometric* PMF

$$p_{l_1}(l) = P(1-P)^{l-1} \qquad l = 1, 2, \ldots$$

$$p_{l_1}{}^T(z) = \frac{zP}{1 - z(1-P)} \qquad E(l_1) = \frac{1}{P} \qquad \sigma_{l_1}{}^2 = \frac{1-P}{P^2}$$

The number of trials up to and including the rth success, l_r, is called the *rth-order interarrival time.* Random variable l_r, the sum of r independent experimental values of l_1, has the *Pascal* PMF

$$p_{l_r}(l) = \binom{l-1}{r-1} P^r(1-P)^{l-r} \qquad l = r, r+1, \ldots ; \quad r = 1, 2, \ldots$$

$$p_{l_r}^T(z) = \left[\frac{zP}{1-z(1-P)}\right]^r \qquad E(l_r) = \frac{r}{P} \qquad \sigma_{l_r}^2 = \frac{r(1-P)}{P^2}$$

We conclude with one useful observation based on the definition of the Bernoulli process. *Any* events, defined on nonoverlapping sets of trials, are independent. If we have a list of events defined on a series of Bernoulli trials, but there is no trial whose outcome is relevant to the occurrence or nonoccurrence of more than one event in the list, the events in the list are mutually independent. This result, of course, is due to the independence of the individual trials and is often of value in the solution of problems.

4-4 An Example

We consider one example of the application of our results for the Bernoulli process. The first five parts are a simple drill, and part (*f*) will lead us into a more interesting discussion.

Fred is giving out samples of dog food. He makes calls door to door, but he leaves a sample (one can) only on those calls for which the door is answered *and* a dog is in residence. On any call the probability of the door being answered is 3/4, and the probability that any household has a dog is 2/3. Assume that the events "Door answered" and "A dog lives here" are independent and also that the outcomes of all calls are independent.

(a) Determine the probability that Fred gives away his first sample on his third call.

(b) Given that he has given away exactly four samples on his first eight calls, determine the conditional probability that Fred will give away his fifth sample on his eleventh call.

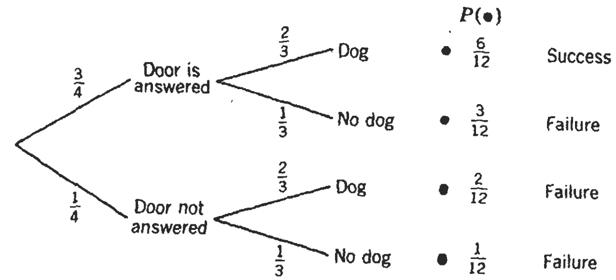
(c) Determine the probability that he gives away his second sample on his fifth call.

(d) Given that he did not give away his second sample on his second call, determine the conditional probability that he will leave his second sample on his fifth call.

(e) We shall say that Fred "needs a new supply" immediately *after* the call on which he gives away his last can. If he starts out with two cans, determine the probability that he completes at least five calls before he needs a new supply.

(f) If he starts out with exactly m cans, determine the expected value and variance of d_m, the number of homes with dogs which he passes up (because of no answer) before he needs a new supply.

We begin by sketching the event space for each call.



For all but the last part of this problem, we may consider each call to be a Bernoulli trial where the probability of success (door answered and dog in residence) is given by $P = \frac{3}{4} \cdot \frac{2}{3} = \frac{1}{2}$.

a Fred will give away his first sample on the third call if the first two calls are failures and the third is a success. Since the trials are independent, the probability of this sequence of events is simply $(1-P)(1-P)P = 1/8$. Another way to obtain this answer is to realize that, in the notation of the previous section, we want $p_{l_1}(3)$ which is $(1-P)^2P = 1/8$.

b The event of interest requires failures on the ninth and tenth trials and a success on the eleventh trial. For a Bernoulli process, the outcomes of these three trials are independent of the results of any other trials, and again our answer is $(1-P)(1-P)P = 1/8$.

c We desire the probability that l_2, the second-order interarrival time, is equal to five trials. We know that $p_{l_2}(l)$ is a Pascal PMF, and we have

$$p_{l_2}(5) = \binom{5-1}{2-1} P^2(1-P)^{5-2} = \frac{4}{32} = \frac{1}{8}$$

d Here we require the conditional probability that the experimental value of l_2 is equal to 5, given that it is greater than 2.

$$p_{l_2|l_2>2}(5 \mid l_2 > 2) = \frac{p_{l_2}(5)}{\text{Prob}(l_2 > 2)} = \frac{p_{l_2}(5)}{1 - p_{l_2}(2)} \qquad l = 3, 4, 5, \ldots$$

$$= \frac{\binom{5-1}{2-1} P^2(1-P)^{5-2}}{1 - \binom{2-1}{2-1} P^2(1-P)^0} = \frac{1/8}{3/4} = \frac{1}{6}$$

As we would expect, by excluding the possibility of one particular experimental value of l_2, we have increased the probability that the experimental value of l_2 is equal to 5. The PMF for the total number of trials up to and including the rth success (since the process began) does, of course, depend on the past history of the process.

e The probability that Fred will complete at least five calls before he needs a new supply is equal to the probability that the experimental value of l_2 is greater than or equal to 5.

$$\text{Prob}(l_2 \geq 5) = 1 - \text{Prob}(l_2 \leq 4) = 1 - \sum_{l=2}^{4} \binom{l-1}{2-1} P^2(1-P)^{l-2} = \frac{5}{16}$$

f Let discrete random variable f_m represent the number of failures before Fred runs out of samples on his mth successful call. Since l_m is the number of trials up to and including the mth success, we have $f_m = l_m - m$. Given that Fred makes l_m calls before he needs a new supply, we can regard each of the f_m unsuccessful calls as trials in another Bernoulli process where P', the probability of a success (a disappointed dog), is found from the above event space to be

$$P' = \text{Prob(dog lives there | Fred did not leave a sample)}$$
$$= \frac{(1/4)(2/3)}{(3/4)(1/3) + (1/4)(2/3) + (1/4)(1/3)} = \frac{1}{3}$$

We define x to be a Bernoulli random variable with parameter P'.

The number of dogs passed up before Fred runs out, d_m, is equal to the sum of f_m (a random number) Bernoulli random variables each with $P' = 1/3$. From Sec. 3-7, we know that the z transform of $p_{d_m}(d)$ is equal to the z transform of $p_{f_m}(f)$, with z replaced by the z transform of Bernoulli random variable x. Without formally obtaining $p_{d_m}{}^T(z)$, we may use the results of Sec. 3-7 to evaluate $E(d_m)$ and $\sigma_{d_m}{}^2$.

$$E(d_m) = E(f_m)E(x) \qquad \text{from Sec. 3-7}$$

$$E(f_m) = E(l_m - m) = \frac{m}{P} - m = m\frac{1-P}{P} \qquad E(x) = P'$$

We substitute these expected values into the above equation for $E(d_m)$, the expected value of the number of dogs passed up.

$$E(d_m) = m\frac{1-P}{P}P' = m\frac{\frac{1}{2}}{\frac{1}{2}}\frac{1}{3} = \frac{m}{3} = \text{expected value of no. of dogs passed up before Fred gives away } m\text{th sample}$$

We find the variance of d_m by

$$\sigma_{d_m}{}^2 = E(f_m)\sigma_x{}^2 + [E(x)]^2\sigma_{f_m}{}^2 \qquad \text{from Sec. 3-7}$$

$$\sigma_{f_m}{}^2 = \sigma_{l_m}{}^2$$

Since $f_m = l_m - m$, the PMF for f_m is simply the PMF for l_m shifted to the left by m. Such a shift doesn't affect the spread of the PMF about its expected value.

$$\sigma_{l_m}{}^2 = m\frac{(1-P)}{P^2} \qquad \text{from properties of Pascal PMF noted in previous section}$$

We may now substitute into the above equation for $\sigma_{d_m}{}^2$, the variance of the number of dogs passed up.

$$\sigma_{d_m}{}^2 = m\frac{1-P}{P}P'(1-P') + (P')^2 m\frac{1-P}{P^2}$$
$$= m\frac{\frac{1}{2}}{\frac{1}{2}}\cdot\frac{1}{3}\cdot\frac{2}{3} + \left(\frac{1}{3}\right)^2 m\frac{\frac{1}{2}}{\frac{1}{4}} = \frac{4m}{9}$$

Although the problem did not require it, let's obtain the z transform of $p_{d_m}(d)$, which is to be obtained by substitution into

$$p_{f_m}{}^T[p_x{}^T(z)]$$

We know that $p_x{}^T(z) = 1 - P' + P'z = \frac{2}{3} + \frac{1}{3}z$, and, using the fact that $f_m = l_m - m$, we can write out $p_{l_m}{}^T(z)$ and $p_{f_m}{}^T(z)$ and note a simple relation to obtain the latter from the former.

$$p_{l_m}{}^T(z) = p_{l_m}(m)z^m + p_{l_m}(m+1)z^{m+1} + p_{l_m}(m+2)z^{m+2} + \cdots$$
$$p_{f_m}{}^T(z) = p_{l_m}(m)z^0 + p_{l_m}(m+1)z^1 + p_{l_m}(m+2)z^2 + \cdots$$

From these expansions and our results from the Pascal process we have

$$p_{f_m}{}^T(z) = z^{-m}p_{l_m}{}^T(z) = P^m[1 - z(1-P)]^{-m}$$

and, finally,

$$p_{d_m}{}^T(z) = p_{f_m}{}^T[p_x{}^T(z)] = P^m[1 - (1 - P' + zP')(1-P)]^{-m}$$
$$= \left(\frac{1}{2}\right)^m\left(\frac{2}{3} - \frac{z}{6}\right)^{-m}$$

Since the z transform for the PMF of the number of dogs passed up happened to come out in such a simple form, we can find the PMF $p_{d_m}(d)$ by applying the inverse transform relationship from Sec. 3-2. We omit the algebraic work and present the final form of $p_{d_m}(d)$.

$$p_{d_m}(d) = \frac{1}{d!}\left[\frac{d^d}{dz^d}p_{d_m}{}^T(z)\right]_{z=0}$$
$$= \binom{d+m-1}{d}\left(\frac{3}{4}\right)^m\left(\frac{1}{4}\right)^d \qquad m = 1, 2, 3, \ldots; \quad d = 0, 1, 2, \ldots$$

For instance, if Fred starts out with only one sample, we have $m = 1$ and

$$p_{d_1}(d) = (\tfrac{3}{4})(\tfrac{1}{4})^d \qquad d = 0, 1, 2, \ldots$$

is the PMF for the number of dogs who were passed up (Fred called but door not answered) while Fred was out making calls to try and give away his one sample.

4-5 The Poisson Process

We defined the Bernoulli process by a particular probabilistic description of the "arrivals" of successes in a series of independent identical discrete trials. The Poisson process will be defined by a probabilistic description of the behavior of arrivals at *points* on a continuous line.

For convenience, we shall generally refer to this line as if it were a time (t) axis. From the definition of the process, we shall see that a Poisson process may be considered to be the limit, as $\Delta t \to 0$ of a series of identical independent Bernoulli trials, one every Δt, with the probability of a success on any trial given by $P = \lambda\, \Delta t$.

For our study of the Poisson process we shall adopt the somewhat improper notation:

$\mathcal{P}(k,t)$ = the probability that there are exactly k arrivals during any interval of duration t

This notation, while not in keeping with our more aesthetic habits developed earlier, is compact and particularly convenient for the types of equations to follow. We observe that $\mathcal{P}(k,t)$ is a PMF for random variable k for any fixed value of parameter t. In any interval of length t, with $t \geq 0$, we must have exactly zero, or exactly one, or exactly two, etc., arrivals. Thus we have

$$\sum_{k=0}^{\infty} \mathcal{P}(k,t) = 1$$

We also note that $\mathcal{P}(k,t)$ is *not* a PDF for t. Since $\mathcal{P}(k,t_1)$ and $\mathcal{P}(k,t_2)$ are not mutually exclusive events, we can state only that

$$0 \leq \int_{t=0}^{\infty} \mathcal{P}(k,t)\, dt \leq \infty$$

The use of random variable k to count arrivals is consistent with our notation for counting successes in a Bernoulli process.

There are several equivalent ways to define the Poisson process. We shall define it directly in terms of those properties which are most useful for the analysis of problems based on physical situations.

Our definition of the Poisson process is as follows:

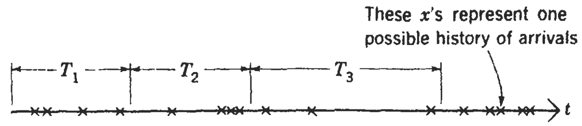
1 *Any* events defined on nonoverlapping time intervals are mutually independent.

2 The following statements are correct for suitably small values of Δt:

$$\mathcal{P}(k,\Delta t) = \begin{cases} 1 - \lambda\, \Delta t & k = 0 \\ \lambda\, \Delta t & k = 1 \\ 0 & k > 1 \end{cases}$$

The first of the above two defining properties establishes the no-memory attribute of the Poisson process. As an example, for a

Poisson process, events A, B, and C, defined on the intervals shown below, are mutually independent.



Event A: Exactly k_1 arrivals in interval T_1 and exactly k_3 arrivals in interval T_3
Event B: More than k_2 arrivals in interval T_2
Event C: No arrivals in the hour which begins 10 minutes after the third arrival following the end of interval T_3

The second defining property for the Poisson process states that, *for small enough intervals*, the probability of having exactly one arrival within one such interval is proportional to the duration of the interval and that, to the first order, the probability of more than one arrival within one such interval is zero. This simply means that $\mathcal{P}(k,\Delta t)$ can be expanded in a Taylor series about $\Delta t = 0$, and when we neglect terms of order $(\Delta t)^2$ or higher, we obtain the given expressions for $\mathcal{P}(k,\Delta t)$.

Among other things, we wish to determine the expression for $\mathcal{P}(k,t)$ for $t \geq 0$ and for $k = 0, 1, 2, \ldots$. Before doing the actual derivation, let's reason out how we would expect the result to behave. From the definition of the Poisson process and our interpretation of it as a series of Bernoulli trials in incremental intervals, we expect that

$\mathcal{P}(0,t)$ as a function of t should be unity at $t = 0$ and decrease monotonically toward zero as t increases. (The event of exactly zero arrivals in an interval of length t requires more and more successive failures in incremental intervals as t increases.)

$\mathcal{P}(k,t)$ as a function of t, for $k > 0$, should start out at zero for $t = 0$, increase for a while, and then decrease toward zero as t gets very large. [The probability of having exactly k arrivals (with $k > 0$) should be very small for intervals which are too long or too short.]

$\mathcal{P}(k,0)$ as a function of k should be a bar graph with only one nonzero bar; there will be a bar of height unity at $k = 0$.

We shall use the defining properties of the Poisson process to relate $\mathcal{P}(k, t + \Delta t)$ to $\mathcal{P}(k,t)$ and then solve the resulting differential equations to obtain $\mathcal{P}(k,t)$.

For a Poisson process, if Δt is small enough, we need consider only the possibility of zero or one arrivals between t and $t + \Delta t$. Taking advantage also of the independence of events in nonoverlapping time intervals, we may write

$$\mathcal{P}(k, t + \Delta t) = \mathcal{P}(k,t)\mathcal{P}(0,\Delta t) + \mathcal{P}(k-1, t)\mathcal{P}(1,\Delta t)$$

The two terms summed on the right-hand side are the probabilities of the only two (mutually exclusive) histories of the process which may lead to having exactly k arrivals in an interval of duration $t + \Delta t$. Our definition of the process specified $\mathcal{P}(0,\Delta t)$ and $\mathcal{P}(1,\Delta t)$ for small enough Δt. We substitute for these quantities to obtain

$$\mathcal{P}(k, t + \Delta t) = \mathcal{P}(k,t)(1 - \lambda\,\Delta t) + \mathcal{P}(k-1, t)\lambda\,\Delta t$$

Collecting terms, dividing through by Δt, and taking the limit as $\Delta t \to 0$, we find

$$\frac{d}{dt}\mathcal{P}(k,t) + \lambda\mathcal{P}(k,t) = \lambda\mathcal{P}(k-1, t)$$

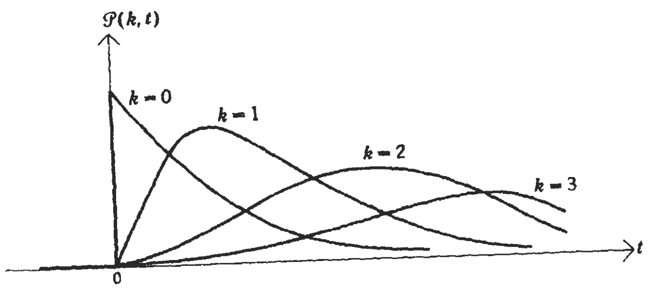
which may be solved iteratively for $k = 0$ and then for $k = 1$, etc., subject to the initial conditions

$$\mathcal{P}(k,0) = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases}$$

The solution for $\mathcal{P}(k,t)$, which may be verified by direct substitution, is

$$\mathcal{P}(k,t) = \frac{(\lambda t)^k e^{-\lambda t}}{k!} \qquad t \geq 0; \quad k = 0, 1, 2, \ldots$$

And we find that $\mathcal{P}(k,t)$ does have the properties we anticipated earlier.



Letting $\mu = \lambda t$, we may write this result in the more proper notation for a PMF as

$$p_k(k_0) = \frac{(\lambda t)^{k_0} e^{-\lambda t}}{k_0!} = \frac{\mu^{k_0} e^{-\mu}}{k_0!} \qquad \mu = \lambda t; \quad k_0 = 0, 1, 2, \ldots$$

This is known as the *Poisson* PMF. Although we derived the Poisson PMF by considering the number of arrivals in an interval of length t for a certain process, this PMF arises frequently in many other situations.

To obtain the expected value and variance of the Poisson PMF,

we'll use the z transform

$$p_k^T(z) = \sum_{k_0=0}^{\infty} p_k(k_0) z^{k_0} = e^{-\mu} \sum_{k_0=0}^{\infty} \frac{(\mu z)^{k_0}}{k_0!} = e^{\mu(z-1)}$$

$$E(k) = \left[\frac{d}{dz} p_k^T(z)\right]_{z=1} = \mu$$

$$\sigma_k^2 = \left\{\frac{d^2}{dz^2} p_k^T(z) + \frac{d}{dz} p_k^T(z) - \left[\frac{d}{dz} p_k^T(z)\right]^2\right\}_{z=1} = \mu$$

Thus the expected value and variance of Poisson random variable k are both equal to μ.

We may also note that, since $E(k) = \lambda t$, we have an interpretation of the constant λ used in

$$\mathcal{P}(k,\Delta t) = \begin{cases} 1 - \lambda\,\Delta t & k = 0 \\ \lambda\,\Delta t & k = 1 \\ 0 & k = 2, 3, \ldots \end{cases}$$

as part of the definition of the Poisson process. The relation $E(k) = \lambda t$ indicates that λ is the expected number of arrivals per unit time in a Poisson process. The constant λ is referred to as the *average arrival rate* for the process.

Incidentally, another way to obtain $E(k) = \lambda t$ is to realize that, for sufficiently short increments, the expected number of arrivals in a time increment of length Δt is equal to $0 \cdot (1 - \lambda\,\Delta t) + 1 \cdot \lambda\,\Delta t = \lambda\,\Delta t$. Since an interval of length t is the sum of $t/\Delta t$ such increments, we may determine $E(k)$ by summing the expected number of arrivals in each such increment. This leads to $E(k) = \lambda\,\Delta t \cdot \dfrac{t}{\Delta t} = \lambda t$.

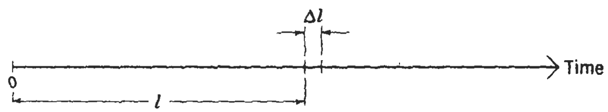
4-6 Interarrival Times for the Poisson Process

Let l_r be a continuous random variable defined to be the interval of time between any arrival in a Poisson process and the rth arrival after it. Continuous random variable l_r, the *rth-order interarrival time*, has the same interpretation here as discrete random variable l_r had for the Bernoulli process.

We wish to determine the PDF's

$$f_{l_r}(l) \qquad l \geq 0; \quad r = 1, 2, 3, \ldots$$

And we again use an argument similar to that for the derivation of the Pascal PMF,



For small enough Δl we may write

$$\text{Prob}(l < l_r \le l + \Delta l) = f_{l_r}(l)\,\Delta l$$

$$f_{l_r}(l)\,\Delta l = \underbrace{\mathcal{P}(r-1, l)}_{A}\,\underbrace{\lambda\,\Delta l}_{B} = \frac{(\lambda l)^{r-1}e^{-\lambda l}}{(r-1)!}\lambda\,\Delta l \qquad l \ge 0; \quad r = 1, 2, \ldots$$

where A = probability that there are exactly $r - 1$ arrivals in an interval of duration l

B = conditional probability that rth arrival occurs in next Δl, given exactly $r - 1$ arrivals in previous interval of duration l

Thus we have obtained the PDF for the rth-order interarrival time

$$f_{l_r}(l) = \frac{\lambda^r l^{r-1}e^{-\lambda l}}{(r-1)!} \qquad l \ge 0; \quad r = 1, 2, \ldots$$

which is known as the *Erlang* family of PDF's. (Random variable l_r is said to be an *Erlang random variable of order r*.)

The first-order interarrival times, described by random variable l_1, have the PDF

$$f_{l_1}(l) = \mu_{-1}(l - 0)\lambda e^{-\lambda l}$$

which is the *exponential* PDF. We may obtain its mean and variance by use of the s transform.

$$f_{l_1}{}^T(s) = \int_{l=-\infty}^{\infty} e^{-sl} f_{l_1}(l)\,dl = \frac{\lambda}{s+\lambda}$$

$$E(l_1) = -\left[\frac{d}{ds} f_{l_1}{}^T(s)\right]_{s=0} = \frac{1}{\lambda}$$

$$\sigma_{l_1}{}^2 = \left\{\frac{d^2}{ds^2} f_{l_1}{}^T(s) - \left[\frac{d}{ds} f_{l_1}{}^T(s)\right]^2\right\}_{s=0} = \frac{1}{\lambda^2}$$

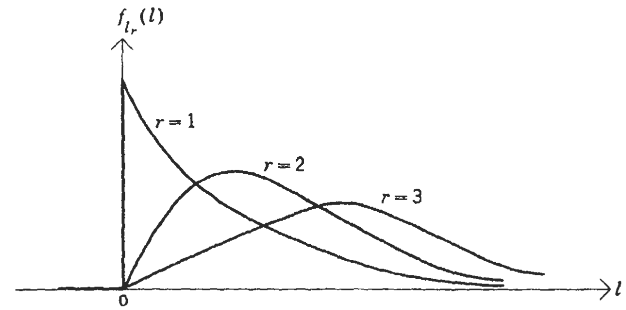
Suppose we are told that it has been τ units of time since the last arrival and we wish to determine the conditional PDF for the duration of the remainder $(l_1 - \tau)$ of the present interarrival time. By conditioning the event space for l_1, we would learn that the PDF for the remaining time until the next arrival is still an exponential random variable with parameter λ (see Prob. 4.06). This result is due to the no-memory (independence of events in nonoverlapping intervals) property of the Poisson process; we discussed a similar result for the Bernoulli process in Sec. 4-2.

Random variable l_r is the sum of r independent experimental values of random variable l_1. Therefore we have

$$E(l_r) = rE(l_1) = \frac{r}{\lambda} \qquad \sigma_{l_r}{}^2 = r\sigma_{l_1}{}^2 = \frac{r}{\lambda^2}$$

$$f_{l_r}{}^T(s) = [f_{l_1}{}^T(s)]^r = \left(\frac{\lambda}{s+\lambda}\right)^r$$

The following is a sketch of some members of Erlang family of PDF's:



We established that the first-order interarrival times for a Poisson process are exponentially distributed mutually independent random variables. Had we taken this to be our definition of the Poisson process, we would have arrived at identical results. The usual way of determining whether it is reasonable to model a physical process as a Poisson process involves checking whether or not the first-order interarrival times are approximately independent exponential random variables.

Finally, we realize that the relation

$$\sum_{r=1}^{\infty} f_{l_r}(l)\,dl = \lambda\,dl \qquad l \ge 0$$

holds for reasons similar to those discussed at the end of Sec. 4-2.

4-7 Some Additional Properties of Poisson Processes and Poisson Random Variables

Before summarizing our results for the Poisson process, we wish to note a few additional properties.

Consider discrete random variable w, the sum of two *independent* Poisson random variables x and y, with expected values $E(x)$ and $E(y)$. There are at least three ways to establish that $p_w(w_0)$ is also a Poisson PMF. One method involves direct summation in the x_0, y_0 event space (see Prob. 2.03). Or we may use z transforms as follows,

$$p_x{}^T(z) = e^{E(x)(z-1)} \qquad p_y{}^T(z) = e^{E(y)(z-1)}$$

$$w = x + y \qquad x, y \text{ independent}$$

$$p_w{}^T(z) = p_x{}^T(z)p_y{}^T(z) = e^{[E(x)+E(y)](z-1)}$$

which we recognize to be the z transform of the Poisson PMF

$$p_w(w_0) = \frac{[E(x) + E(y)]^{w_0}e^{-[E(x)+E(y)]}}{w_0!} \qquad w_0 = 0, 1, \ldots$$

A third way would be to note that $w = x + y$ could represent the total number of arrivals for two independent Poisson processes within a certain interval. A new process which contains the arrivals due to both of the original processes would still satisfy our definition of the Poisson process with $\lambda = \lambda_1 + \lambda_2$ and would generate experimental values of random variable w for the total number of arrivals within the given interval.

We have learned that the arrival process representing all the arrivals in several independent Poisson processes is also Poisson.

Furthermore, suppose that a new arrival process is formed by performing an independent Bernoulli trial for each arrival in a Poisson process. With probability P, any arrival in the Poisson process is also considered an arrival at the same time in the new process. With probability $1 - P$, any particular arrival in the original process does not appear in the new process. The new process formed in this manner (by "independent random erasures") still satisfies the definition of a Poisson process and has an average arrival rate equal to λP and the expected value of the first-order interarrival time is equal to $(\lambda P)^{-1}$.

If the erasures are not independent, then the derived process has memory. For instance, if we erase alternate arrivals in a Poisson process, the remaining arrivals do not form a Poisson process. It is clear that the resulting process violates the definition of the Poisson process, since, given that an arrival in the new process just occurred, the probability of another arrival in the new process in the next Δt is zero (this would require two arrivals in Δt in the underlying Poisson process). This particular derived process is called an *Erlang process* since the first-order interarrival times are independent and have (second-order) Erlang PDF's. This derived process is one example of how we can use the memoryless Poisson process to model more involved situations with memory.

4-8 Summary of the Poisson Process

For convenience, assume that we are concerned with arrivals which occur at points on a continuous time axis. Quantity $\mathcal{P}(k,t)$ is defined to be the probability that any interval of duration t will contain exactly k arrivals. A process is said to be a *Poisson* process if and only if

1 *For suitably small Δt, $\mathcal{P}(k,\Delta t)$ satisfies*

$$\mathcal{P}(k,\Delta t) = \begin{cases} 1 - \lambda\,\Delta t & k = 0 \\ \lambda\,\Delta t & k = 1 \\ 0 & k > 1 \end{cases}$$

2 *Any events defined on nonoverlapping intervals of time are mutually independent.*

An alternative definition of a Poisson process is the statement that the first-order interarrival times be independent identically distributed exponential random variables.

Random variable k, the number of arrivals in an interval of duration t, is described by the *Poisson* PMF

$$p_k(k_0) = \frac{(\lambda t)^{k_0} e^{-\lambda t}}{k_0!} \qquad t \geq 0; \quad k_0 = 0, 1, 2, \ldots$$

$$p_k^T(z) = e^{\lambda t(z-1)} \qquad E(k) = \lambda t \qquad \sigma_k^2 = \lambda t$$

The first-order interarrival time l_1 is an *exponential* random variable with the PDF

$$f_{l_1}(l) = \lambda e^{-\lambda l} \qquad l \geq 0$$

$$f_{l_1}^T(s) = \frac{\lambda}{s + \lambda} \qquad E(l_1) = \frac{1}{\lambda} \qquad \sigma_{l_1}^2 = \frac{1}{\lambda^2}$$

The time until the rth arrival, l_r, is known as the *rth-order waiting time*, is the sum of r independent experimental values of l_1, and is described by the *Erlang* PDF

$$f_{l_r}(l) = \frac{\lambda^r l^{r-1} e^{-\lambda l}}{(r-1)!} \qquad l \geq 0; \quad r = 1, 2, \ldots$$

$$f_{l_r}^T(s) = \left(\frac{\lambda}{s + \lambda}\right)^r \qquad E(l_r) = rE(l_1) = \frac{r}{\lambda} \qquad \sigma_{l_r}^2 = r\sigma_{l_1}^2 = \frac{r}{\lambda^2}$$

The sum of several independent Poisson random variables is also a random variable described by a Poisson PMF. If we form a new process by including all arrivals due to several independent Poisson processes, the new process is also Poisson. If we perform Bernoulli trials to make independent random erasures from a Poisson process, the remaining arrivals also form a Poisson process.

4-9 Examples

The Poisson process finds wide application in the modeling of probabilistic systems. We begin with a simple example and proceed to consider some rather structured situations. Whenever it seems informative, we shall solve these problems in several ways.

example 1 The PDF for the duration of the (independent) interarrival times between successive cars on the Trans-Australian Highway is given by

$$f_t(t_0) = \begin{cases} \frac{1}{12}e^{-t_0/12} & t_0 \geq 0 \\ 0 & t_0 < 0 \end{cases}$$

where these durations are measured in seconds.

(a) An old wombat requires 12 seconds to cross the highway, and he starts out immediately after a car goes by. What is the probability that he will survive?

(b) Another old wombat, slower but tougher, requires 24 seconds to cross the road, but it takes two cars to kill him. (A single car won't even slow him down.) If he starts out at a random time, determine the probability that he survives.

(c) If both these wombats leave at the same time, immediately after a car goes by, what is the probability that exactly one of them survives?

a Since we are given that the first-order interarrival times are independent exponentially distributed random variables, we know that the vehicle arrivals are Poisson, with

$$\mathcal{P}(k,t) = \frac{(t/12)^k e^{-t/12}}{k!} \qquad k = 0, 1, 2, \ldots ; \quad t \geq 0$$

Since the car-arrival process is memoryless, the time since the most recent car went by until the wombat starts to cross is irrelevant. The fast wombat will survive only if there are exactly zero arrivals in the first 12 seconds after he starts to cross.

$$\mathcal{P}(0,12) = \frac{1^0 e^{-1}}{0!} = e^{-1} = 0.368$$

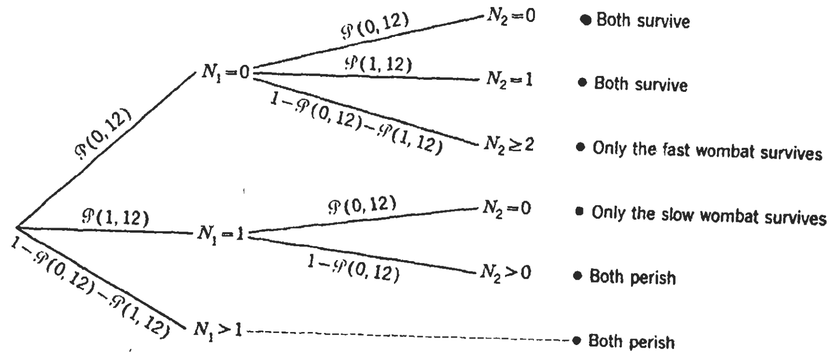
Of course, this must be the same as the probability that the wait until the next arrival is longer than 12 seconds.

$$\mathcal{P}(0,12) = \int_{t=12}^{\infty} \tfrac{1}{12} e^{-t/12}\, dt = 0.368$$

b The slower but tougher wombat will survive only if there is exactly zero or one car in the first 24 seconds after he starts to cross.

$$\mathcal{P}(0,24) + \mathcal{P}(1,24) = \frac{2^0 e^{-2}}{0!} + \frac{2^1 e^{-2}}{1!} = 3e^{-2} = 0.406$$

c Let $\begin{Bmatrix} N_1 \\ N_2 \end{Bmatrix}$ be the number of cars in the $\begin{Bmatrix} \text{first} \\ \text{second} \end{Bmatrix}$ 12 seconds after the wombats start out. It will be helpful to draw a sequential event space for the experiment.



$$\text{Prob(exactly one wombat survives)} = \text{Prob}(N_1 = 0, N_2 \geq 2) + \text{Prob}(N_1 = 1, N_2 = 0)$$

Quantities N_1 and N_2 are independent random variables because they are defined on nonoverlapping intervals of a Poisson process. We may now collect the probability of exactly one survival from the above event space.

$$\text{Prob(exactly 1 wombat survives)} = \mathcal{P}(0,12)[1 - \mathcal{P}(0,12) - \mathcal{P}(1,12)] + \mathcal{P}(1,12)\mathcal{P}(0,12)$$

$$\text{Prob(exactly 1 wombat survives)} = e^{-1}(1 - 2e^{-1}) + e^{-2} = e^{-1} - e^{-2} = 0.233$$

example 2 Eight light bulbs are turned on at $t = 0$. The lifetime of any particular bulb is independent of the lifetimes of all other bulbs and is described by the PDF

$$f_t(t_0) = \begin{cases} \lambda e^{-\lambda t_0} & \text{if } t_0 \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Determine the mean, variance, and s transform of random variable y, the time until the third failure.

We define t_{ij} to be a random variable representing the time from the ith failure until the jth failure, where t_{01} is the duration from $t = 0$ until the first failure. We may write

$$y = t_{03} = t_{01} + t_{12} + t_{23}$$

The length of the time interval during which exactly $8 - i$ bulbs are on is equal to $t_{i(i+1)}$. While $8 - i$ bulbs are on, we are dealing with the sum of $8 - i$ independent Poisson processes and the probability of a failure in the next Δt is equal to $(8 - i)\lambda\,\Delta t$. Thus, from the properties of the Poisson process, we have

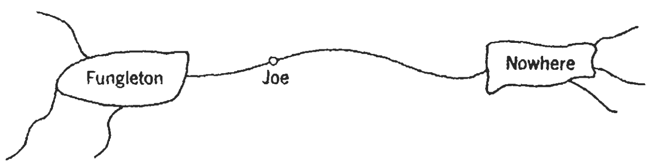
$$E(y) = E(t_{01}) + E(t_{12}) + E(t_{23}) = \frac{1}{8\lambda} + \frac{1}{7\lambda} + \frac{1}{6\lambda}$$

Knowledge of the experimental value of, for instance, t_{01} does not tell us anything about t_{12}. Random variable t_{12} would still be an exponential random variable representing the time until the next arrival for a Poisson process with an average arrival rate of 7λ. Random variables t_{01}, t_{12}, and t_{23} are mutually independent (why?), and we have

$$\sigma_y^2 = \sigma_{t_{01}}^2 + \sigma_{t_{12}}^2 + \sigma_{t_{23}}^2 = \frac{1}{(8\lambda)^2} + \frac{1}{(7\lambda)^2} + \frac{1}{(6\lambda)^2}$$

$$f_y^T(s) = f_{t_{01}}^T(s) \times f_{t_{12}}^T(s) \times f_{t_{23}}^T(s) = \frac{8\lambda}{s + 8\lambda}\,\frac{7\lambda}{s + 7\lambda}\,\frac{6\lambda}{s + 6\lambda}$$

This has been one example of how easily we can obtain answers for many questions related to Poisson models. A harder way to go about it would be to determine first the PDF for y, the third smallest of eight independent identically distributed exponential random variables.



example 3 Joe is waiting for a Nowhere-to-Fungleton (NF) bus, and he knows that, out where he is, arrivals of $\begin{Bmatrix} FN \\ NF \end{Bmatrix}$ buses may be considered independent Poisson processes with average arrival rates of $\begin{Bmatrix} \lambda_{FN} \\ \lambda_{NF} \end{Bmatrix}$ buses per hour. Determine the PMF and the expectation for random variable K, the number of "wrong-way" buses he will see arrive before he boards the next NF bus.

We shall do this problem in several ways.

Method A

We shall obtain the compound PDF for the amount of time he waits and the number of wrong-way buses he sees. Then we determine $p_K(K_0)$ by integrating out over the other random variable. We know the marginal PDF for his waiting time, and it is simple to find the PMF for K conditional on his waiting time. The product of these probabilities tells us all there is to know about the random variables of interest.

The time Joe waits until the first right-way (NF) bus is simply

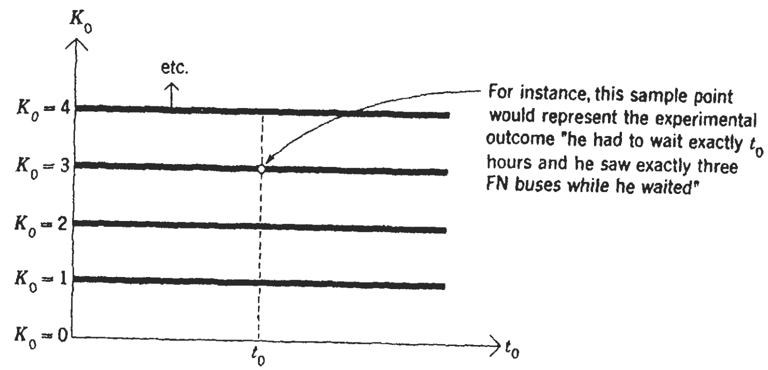
the waiting time until the first arrival in a Poisson process with average arrival rate λ_{NF}. The probability that his total waiting time t will be between t_0 and $t_0 + dt_0$ is

$$f_t(t_0)\,dt_0 = \lambda_{NF}e^{-\lambda_{NF}t_0}\,dt_0$$

Given that the experimental value of Joe's waiting time is exactly t_0 hours, the conditional PMF for K is simply the probability of exactly K_0 arrivals in an interval of duration t_0 for a Poisson process with average arrival rate λ_{FN}.

$$p_{K|t}(K_0 \mid t_0) = \frac{(\lambda_{FN}t_0)^{K_0}e^{-\lambda_{FN}t_0}}{K_0!} \qquad K_0 = 0, 1, 2, \ldots$$

The experiment of Joe's waiting for the next NF bus and observing the number of wrong-way buses while he waits has a two-dimensional event space which is discrete in K and continuous in t.



We obtain the probability assignment in this event space, $f_{t,k}(t_0,k_0)$.

$$f_{t,K}(t_0,K_0) = f_t(t_0)p_{K|t}(K_0 \mid t_0)$$

$$= \frac{\lambda_{NF}e^{-\lambda_{NF}t_0}(\lambda_{FN}t_0)^{K_0}e^{-\lambda_{FN}t_0}}{K_0!} \qquad t_0 \geq 0; \quad K_0 = 0, 1, 2, \ldots$$

The marginal PMF $p_K(K_0)$ may be found from

$$p_K(K_0) = \int_{t_0=0}^{\infty} f_{t,K}(t_0,K_0)\,dt_0 = \frac{\lambda_{NF}\lambda_{FN}^{K_0}}{K_0!}\int_{t_0=0}^{\infty} t_0^{K_0}e^{-(\lambda_{FN}+\lambda_{NF})t_0}\,dt_0$$

By noting that

$$\frac{(\lambda_{FN} + \lambda_{NF})^{K_0+1}t_0^{K_0}e^{-(\lambda_{FN}+\lambda_{NF})t_0}}{K_0!}$$

would integrate to unity over the range $0 \leq t_0 \leq \infty$ (since it is an Erlang PDF of order $K_0 + 1$), we can perform the above integration by inspection to obtain (with $\lambda_{NF}/\lambda_{FN} = \rho$),

$$p_K(K_0) = \frac{\rho}{(1+\rho)^{K_0+1}} \qquad K_0 = 0, 1, 2, \ldots$$

If the average arrival rates λ_{NF} and λ_{FN} are equal ($\rho = 1$), we note that the probability that Joe will see a total of exactly K_0 wrong-way buses before he boards the first right-way bus is equal to $(\frac{1}{2})^{K_0+1}$. For this case, the probability is 0.5 that he will see no wrong way buses while he waits.

The expected value of the number of FN buses he will see arrive may be obtained from the z transform.

$$p_K{}^T(z) = \frac{\rho}{1+\rho} \sum_{K_0=0}^{\infty} \frac{z^{K_0}}{(1+\rho)^{K_0}} = \rho(1+\rho-z)^{-1}$$

$$E(K) = \left[\frac{d}{dz} p_K{}^T(z)\right]_{z=1} = \rho^{-1}$$

This answer seems reasonable for the cases $\rho >> 1$ and $\rho << 1$.

Method B

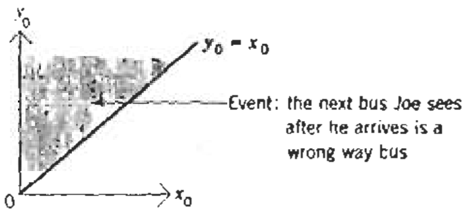
Regardless of when Joe arrives, the probability that the next bus is a wrong bus is simply the probability that an experimental value of a random variable with PDF

$$f_x(x_0) = \lambda_{FN} e^{-\lambda_{FN} x_0} \qquad x_0 \geq 0$$

is smaller than an experimental value of another, independent, random variable with PDF

$$f_y(y_0) = \lambda_{NF} e^{-\lambda_{NF} y_0} \qquad y_0 \geq 0$$

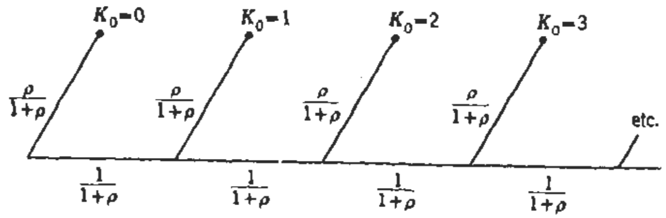
So, working in the x_0, y_0 event space



$$\text{Prob (next bus is an } FN \text{ bus)} = \int_{x_0=0}^{\infty} dx_0 \int_{y_0=x_0}^{\infty} dy_0\, \lambda_{FN}\lambda_{NF} e^{-\lambda_{FN}x_0} e^{-\lambda_{NF}y_0}$$

$$= \frac{\lambda_{FN}}{\lambda_{FN}+\lambda_{NF}} = \frac{1}{1+\rho} \qquad \text{with } \rho = \frac{\lambda_{NF}}{\lambda_{FN}}$$

As soon as the next bus does come, *the same result holds for the following bus;* so we can draw out the sequential event space where each trial corresponds to the arrival of another bus, and the experiment terminates with the arrival of the first NF bus.

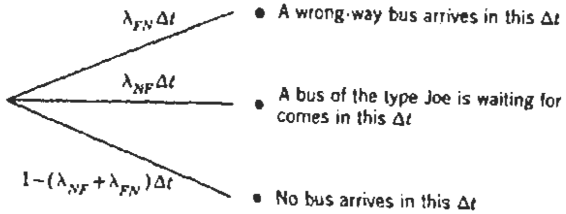


This again leads to

$$p_K(K_0) = \frac{\rho}{(1+\rho)^{K_0+1}} \qquad K_0 = 0, 1, 2, \ldots$$

Method C

Consider the event space for *any* adequately small Δt,



We need be interested in a Δt only if a bus arrives during that Δt; so we may work in a conditional space containing only the upper two event points to obtain

$$\text{Prob (any particular bus is } FN) = \frac{\lambda_{FN}}{\lambda_{FN}+\lambda_{NF}}$$

$$\text{Prob (any particular bus is } NF) = \frac{\lambda_{NF}}{\lambda_{FN}+\lambda_{NF}}$$

This approach replaces the integration in the x,y event space for the previous solution and, of course, leads to the same result.

As a final point, note that N, the *total* number of buses Joe would see if he waited until the Rth NF bus, would have a Pascal PMF. The arrival of each bus would be a Bernoulli trial, and a success is represented by the arrival of an NF bus. Thus, we have

$$p_N(N_0) = \binom{N_0-1}{R-1} \left(\frac{\lambda_{NF}}{\lambda_{FN}+\lambda_{NF}}\right)^R \left(\frac{\lambda_{FN}}{\lambda_{FN}+\lambda_{NF}}\right)^{N_0-R}$$

$$N_0 = R, R+1, \ldots; \quad R = 1, 2, 3, \ldots$$

where N is the total number of buses (including the one he boards) seen by Joe if his policy is to board the Rth right-way bus to arrive after he gets to the bus stop.

4-10 Renewal Processes

Consider a somewhat more general case of a random process in which arrivals occur at points in time. Such a process is known as a *renewal process* if its first-order interarrival times are mutually independent random variables described by the same PDF. The Bernoulli and Poisson processes are two simple examples of the renewal process. In this and the following section, we wish to study a few basic aspects of the general renewal process.

To simplify our discussion, we shall assume in our formal work that the PDF for the first-order interarrival times (*gaps*) $f_x(x_0)$ is a continuous PDF which does not contain any impulses. [A notational change from $f_{l_1}(l)$ to $f_x(x_0)$ will also simplify our work.]

We begin by determining the conditional PDF for the time until the next arrival when we know how long ago the most recent arrival occurred. In the next section, we develop the consequences of beginning to observe a renewal process at a *random* time.

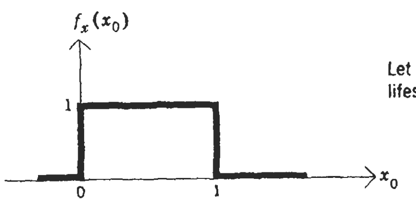
If it is known that the most recent arrival occurred exactly τ units of time ago, application of the definition of conditional probability results in the following conditional PDF for x, the *total* duration of the present interarrival gap:

$$f_{x|x>\tau}(x_0 \mid x > \tau) = \frac{f_x(x_0)}{\int_\tau^\infty f_x(x_0)\,dx_0} = \frac{f_x(x_0)}{1 - p_{x\le}(\tau)} \qquad x_0 > \tau$$

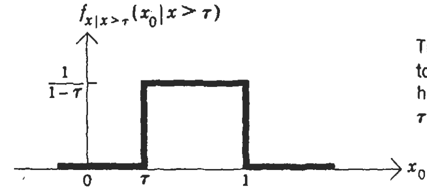
If we let random variable y represent the *remaining* time in the present gap, $y = x - \tau$, we obtain the conditional PDF for y,

$$f_{y|x>\tau}(y_0 \mid x > \tau) = \frac{f_x(y_0 + \tau)}{\int_\tau^\infty f_x(x_0)\,dx_0} = \frac{f_x(y_0 + \tau)}{1 - p_{x\le}(\tau)} \qquad y_0 > 0$$

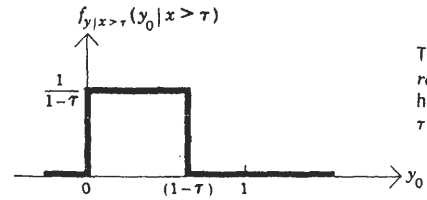
As an example, suppose that we are burning light bulbs one at a time and replacing each bulb the instant it fails. If the lifetimes of the bulbs are independent random variables with PDF $f_x(x_0)$, we have a renewal process in which the points in time at which bulb replacements occur are the arrivals. Let's use the results obtained above to work out one example with a particularly simple form for $f_x(x_0)$.



Let this be the PDF for x, the total lifespan of any individual bulb



This is the conditional PDF for the total lifespan of a bulb, given that it has already been in use for exactly τ units of time without failing



This is the conditional PDF for y, the *remaining* lifespan of a bulb which has already been in use for exactly τ units of time without failing

We learned earlier that the first-order interarrival times for a Poisson process are independent random variables with the PDF $f_x(x_0) = \lambda e^{-\lambda x_0}$ for $x_0 \ge 0$. For a Poisson process we can show by direct substitution that the conditional PDF for the remaining time until the next arrival, $f_{y|x>\tau}(y_0 \mid x > \tau)$, *does not depend on* τ (Prob. 4.06) and is equal to $f_x(y_0)$, the original unconditional PDF for the first-order interarrival times. For the Poisson process (but *not* for the more general renewal process) the time until the next arrival is independent of when we start waiting. If the arrivals of cars at a line across a street constituted a Poisson process, it would be just as safe to start crossing the street at a random time as it would be to start crossing immediately after a car goes by.

4-11 Random Incidence

Assume that a renewal process, characterized by the PDF of its first-order interarrival times, $f_x(x_0)$, has been in progress for a long time. We are now interested in *random incidence*. The relevant experiment is *to pick a time randomly* (for instance, by spinning the hands of a clock) *and then wait until the first arrival in the renewal process after our randomly selected entry time.* The instant of the random entry must always be chosen in a manner which is independent of the actual arrival history of the process.

We wish to determine the PDF for random variable y, the waiting time until the next arrival (or the remaining gap length) following random entry. Several intermediate steps will be required to obtain the unconditional PDF $f_y(y_0)$.

First we shall obtain the PDF for random variable w, the *total* duration of the interarrival gap into which we enter by random incidence. Random variable w describes the duration of an interval which begins with the most recent arrival in the renewal process prior to the instant of random incidence and which terminates with the first arrival in the process after the instant of random incidence.

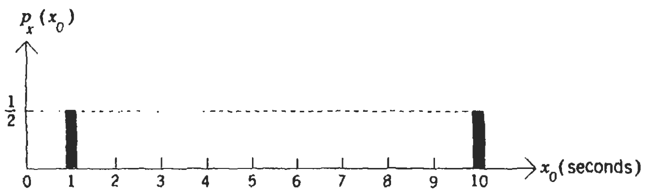
Note that random variables w and x both refer to total interarrival-gap durations for the renewal process, but the experiments on which they are defined are different. An experimental value of w is obtained by determining the total duration of the interarrival gap into which a randomly selected instant falls. An experimental value of x is obtained by noting the duration from any arrival in the renewal process until the next arrival.

After obtaining $f_w(w_0)$, we shall then find the conditional PDF for the remaining time in the gap, y, given the experimental value of the total duration of the gap, w. Thus, our procedure is to work in a w_0,y_0 event space, first obtaining $f_w(w_0)$ and $f_{y|w}(y_0 \mid w_0)$. We then use the relations

$$f_{w,y}(w_0,y_0) = f_w(w_0)f_{y|w}(y_0 \mid w_0) \qquad \text{and} \qquad f_y(y_0) = \int_{w_0} f_{w,y}(w_0,y_0)\, dw_0$$

to obtain the unconditional PDF $f_y(y_0)$ for the waiting time from our randomly selected instant until the next arrival in the renewal process.

To determine the PDF $f_w(w_0)$, let's begin by considering an example where the first-order interarrival times of the renewal process have the discrete PMF



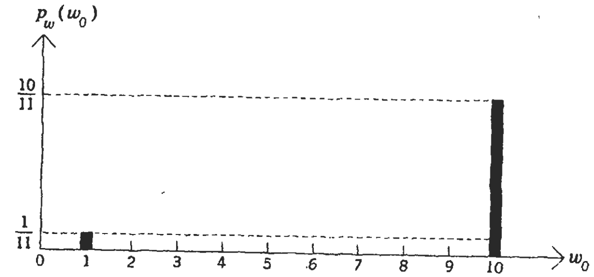
Although any interarrival time is equally likely to be either 1 or 10 seconds long, note that each 10-second gap consumes 10 times as much time as each 1-second gap. The probability that a randomly selected instant of time falls into a 10-second gap is proportional to the fraction of all time which is included in 10-second gaps.

The fraction of all time which is included in gaps of duration x_0 should be, in general, proportional to $p_x(x_0)$ weighted by x_0, since $p_x(x_0)$ is the fraction of the gaps which are of duration x_0 and each such gap consumes x_0 seconds. Recalling that random variable w is to be the total duration of the interarrival gap into which our randomly selected instant falls, we have argued that

$$p_w(w_0) = \frac{w_0 p_x(w_0)}{\sum_{w_0} w_0 p_x(w_0)} = \frac{w_0 p_x(w_0)}{E(x)}$$

where the denominator is the required normalization factor.

For the particular example given above, we obtain the PMF for the total duration of the gap into which a random entry falls,



A random entry, for this example, is ten times as likely to fall into a ten-second gap as a one-second gap, even though a gap length is equally likely to be of either kind.

Extending the general form of $p_w(w_0)$ to the continuous case, we have the desired $f_w(w_0)$

$$f_w(w_0) = \frac{w_0 f_x(w_0)}{\int_{w_0} w_0 f_x(w_0)\, dw_0} = \frac{w_0 f_x(w_0)}{E(x)}$$

where $f_x(\cdot)$ is the PDF for the first-order interarrival times for the renewal process and $f_w(w_0)$ is the PDF for the total duration of the interarrival gap entered by random incidence.

In reasoning our way to this result, we have made certain assumptions about the relation between the probability of an event and the fraction of a large number of trials on which the event will occur. We speculated on the nature of this relation in Sec. 3-6, and the proof will be given in Chap. 6.

Given that we have entered into a gap of total duration w_0 by random incidence, the remaining time in the gap, y, is uniformly distributed between 0 and w_0 with the conditional PDF

$$f_{y|w}(y_0 \mid w_0) = \begin{cases} \dfrac{1}{w_0} & \text{if } 0 \le y_0 \le w_0 \\ 0 & \text{otherwise} \end{cases}$$

because a random instant is as likely to fall within any increment of a w_0-second gap as it is to fall within any other increment of equal duration within the w_0-second gap.

Now we may find the joint PDF for random variables w and y,

$$f_{w,y}(w_0,y_0) = f_w(w_0)f_{y|w}(y_0 \mid w_0) = \frac{w_0 f_x(w_0)}{E(x)} \cdot \frac{1}{w_0} \qquad 0 \le y_0 \le w_0 \le \infty$$

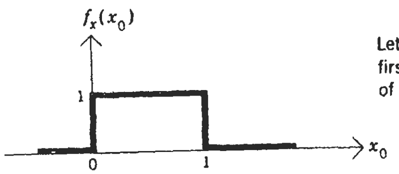
To determine $f_y(y_0)$, the PDF for the time until the next arrival after the instant of random incidence, we need only integrate (carefully) over w_0 in the w_0,y_0 event space. Note that w, the total length of the gap entered by random incidence, must be greater than or equal to y, the remaining time in that gap; so we have

$$f_y(y_0) = \int_{w_0} f_{w,y}(w_0,y_0)\, dw_0 = \int_{w_0=y_0}^{\infty} \frac{f_x(w_0)\, dw_0}{E(x)}$$

$$f_y(y_0) = \frac{1 - p_{x\le}(y_0)}{E(x)}$$

where $f_y(y_0)$ is the PDF for the duration of the interval which begins at a "random" time and terminates with the next arrival for a renewal process with first-order interarrival times described by random variable x.

We apply these results to the problem introduced in the previous section. Let the PDF for the first-order interarrival times be

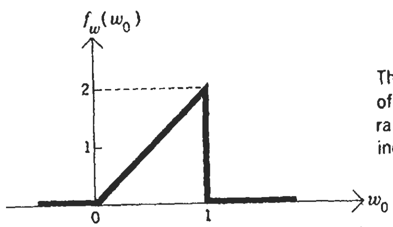


Let this be the PDF for the first-order interarrival times of a renewal process

Now, first we apply the relation

$$f_w(w_0) = \frac{w_0 f_x(w_0)}{E(x)}$$

to obtain

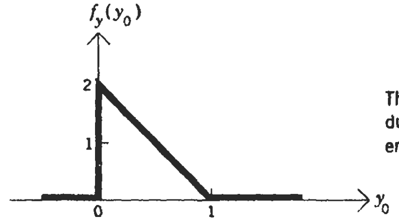


This is the PDF for the total duration of the interarrival gap entered by random incidence. Obviously, random incidence favors entry into longer gaps

and we use

$$f_y(y_0) = \frac{1 - p_{x\le}(y_0)}{E(x)}$$

to obtain



This is the PDF for the remaining duration of the interarrival gap entered by random incidence

It is interesting to note that the expected value of the remaining duration of the gap entered by random incidence, $E(y)$, may be greater than, equal to, or less than the "expected gap length" given by $E(x)$. In fact, we have already learned that $E(x) = E(y)$ for a Poisson process.

Thus, for instance, if car interarrival times are independent random variables described by $f_x(x_0)$, the expected waiting time until the next car arrives may be greater if we start to cross at a random time than if we start right after a car goes by! If we understand the different experiments which give rise to $E(x)$ and $E(y)$, this seems entirely reasonable, since we realize that random incidence favors entry into large gaps.

We should realize that statements about average values or expected values of a random variable are meaningless unless we have a full description of the experiment on whose outcomes the random variable is defined. In the above discussion, $E(x)$ and $E(y)$ are generally different, but each is the "expected value of the time until the next arrival." The experiments of "picking a gap" and "picking an instant of time" may lead to distinctly different results. (Probs. 2.08, 2.11, and 3.10 have already introduced similar concepts.)

PROBLEMS

4.01 The PMF for the number of failures before the rth success in a Bernoulli process is sometimes called the *negative binomial* PMF. Derive it and explain its relation to the Pascal PMF.

4.02 A channel contains a series flow of objects, each of fixed length L. All objects travel at constant velocity V. Each separation S between successive objects is some integral multiple of L, $S = nL$, where the n

for each separation is an independent random variable described by the probability mass function

$$p_n(n_0) = a(1-a)^{n_0-1} \qquad n_0 = 1, 2, 3, \ldots; \quad 0 < a < 1$$

a Find the average flow rate, in objects per unit time, observable at some point in the channel.

b Calculate what additional flow can exist under a rule that the resulting arrangements of objects must have at least a separation of L from adjacent objects.

c As seen by an electric eye across the channel, what fraction of all the gap time is occupied by gaps whose total length is greater than $2L$? A numerical answer is required.

4.03 Let x be a discrete random variable described by a geometric PMF. Given that the experimental value of random variable x is greater than integer y, show that the conditional PMF for $x - y$ is the same as the original PMF for x. Let $r = x - y$, and sketch the following PMF's:

a $p_x(x_0)$ b $p_{x|x>y}(x_0 \mid x > y)$ c $p_r(r_0)$

4.04 We are given two independent Bernoulli processes with parameters P_1 and P_2. A new process is defined to have a success on its kth trial ($k = 1, 2, 3, \ldots$) only if *exactly* one of the other two processes has a success on its kth trial.

a Determine the PMF for the number of trials up to and including the rth success in the new process.

b Is the new process a Bernoulli process?

4.05 Determine the expected value, variance, and z transform for the total number of trials from the start of a Bernoulli process up to and including the nth success after the mth failure.

4.06 Let x be a continuous random variable whose PDF $f_x(x_0)$ contains no impulses. Given that $x > T$, show that the conditional PDF for $r = x - T$ is equal to $f_x(r_0)$ if $f_x(x_0)$ is an exponential PDF.

4.07 To cross a single lane of moving traffic, we require at least a duration T. Successive car interarrival times are independently and identically distributed with probability density function $f_t(t_0)$. If an interval between successive cars is longer than T, we say that the interval represents a single opportunity to cross the lane. Assume that car lengths are small relative to intercar spacing and that our experiment begins the instant after the zeroth car goes by.

Determine, in as simple a form as possible, expressions for the probability that:

a We can cross for the first time just before the Nth car goes by.

b We shall have had exactly n opportunities by the instant the Nth car goes by.

c The occurrence of the nth opportunity is immediately followed by the arrival of the Nth car.

4.08 Consider the manufacture of Grandmother's Fudge Nut Butter Cookies. Grandmother has noted that the number of nuts in a cookie is a random variable with a Poisson mass function and that the average number of nuts per cookie is 1.5.

a What is the numerical value of the probability of having at least one nut in a randomly selected cookie?

b Determine the numerical value of the variance of the number of nuts per cookie.

c Determine the probability that a box of exactly M cookies contains exactly the expected value of the number of nuts for a box of N cookies. ($M = 1, 2, 3, \ldots$; $N = 1, 2, 3, \ldots$)

d What is the probability that a nut selected at random goes into a cookie containing exactly K nuts?

e The customers have been getting restless; so grandmother has instructed her inspectors to discard each cookie which contains exactly zero nuts. Determine the mean and variance of the number of nuts per cookie for the remaining cookies.

4.09 A woman is seated beside a conveyer belt, and her job is to remove certain items from the belt. She has a narrow line of vision and can get these items only when they are right in front of her.

She has noted that the probability that exactly k of her items will arrive in a minute is given by

$$p_k(k_0) = \frac{2^{k_0}e^{-2}}{k_0!} \qquad k_0 = 0, 1, 2, 3, \ldots$$

and she assumes that the arrivals of her items constitute a Poisson process.

a If she wishes to sneak out to have a beer but will not allow the expected value of the number of items she misses to be greater than 5, how much time may she take?

b If she leaves for two minutes, what is the probability that she will miss exactly two items the first minute and exactly one item the second minute?

c If she leaves for two minutes, what is the probability that she will miss a total of exactly three items?

d The union has installed a bell which rings once a minute with precisely one-minute intervals between gongs. If, between two successive gongs, more than three items come along the belt, she will handle only three of them properly and will destroy the rest. Under this system, what is the probability that any particular item will be destroyed?

4.10 Arrivals of certain events at points in time are known to constitute a Poisson process, but it is not known which of two possible values of λ, the average arrival rate, describes the process. Our a priori estimate is that $\lambda = 2$ or $\lambda = 4$ with equal probability.

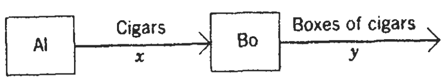
We observe the process for T units of time and observe exactly K arrivals. Given this information, determine the conditional probability that $\lambda = 2$. Check to see whether or not your answer is reasonable for some simple limiting values for K and T.

4.11 Independent experimental values of a geometric random variable are obtained, and we label these values $K_1, K_2, K_3, \ldots$. Random variable r_i is defined by

$$r_i = \sum_{j=1}^{i} K_j \qquad i = 1, 2, \ldots$$

If we eliminate arrivals number $r_1, r_2, r_3, \ldots$ in a Poisson process, do the remaining arrivals constitute a Poisson process?

4.12



Al makes cigars, placing each cigar on a constant-velocity conveyer belt as soon as it is finished. Bo packs the cigars into boxes of four cigars each, placing each box back on the belt as soon as it is filled. The time Al takes to construct any particular cigar is, believe it or not, an independent exponential random variable with an expected value of five minutes.

a Determine $\mathcal{P}_A(k,T)$, the probability that Al makes exactly k cigars in T minutes. Determine the mean and variance of k as a function of T. $k = 0, 1, 2, \ldots;\ 0 \leq T \leq \infty$.

b Determine the probability density function $f_\tau(\tau_0)$, where τ is the interarrival time (measured in minutes) between successive cigars at point x.

c Determine $\mathcal{P}_B(r,T)$, the probability that Bo places exactly r boxes of cigars back on the belt during an interval of T minutes.

d Determine the probability density function $f_t(t_0)$, where t is the interarrival time (measured in minutes) between successive boxes of cigars at point y.

e If we arrive at point y at a random instant, long after the process began, determine the PDF $f_r(r_0)$, where r is the duration of our wait until we see a box of cigars at point y.

4.13 Dave is taking a multiple-choice exam. You may assume that the number of questions is infinite. *Simultaneously, but independently,* his conscious and subconscious facilities are generating answers for him, each in a Poisson manner. (His conscious and subconscious are always working on different questions.)

Average rate at which conscious responses are generated
$= \lambda_c$ responses/min

Average rate at which subconscious responses are generated
$= \lambda_s$ responses/min

Each conscious response is an independent Bernoulli trial with probability p_c of being correct. Similarly, each subconscious response is an independent Bernoulli trial with probability p_s of being correct.

Dave responds only once to each question, and you can assume that his time for recording these conscious and subconscious responses is negligible.

a Determine $p_k(k_0)$, the probability mass function for the number of *conscious responses* Dave makes in an interval of T minutes.

b If we pick any question to which Dave has responded, what is the probability that his answer to that question:

i Represents a conscious response

ii Represents a conscious correct response

c If we pick an interval of T minutes, what is the probability that in that interval Dave will make exactly R_0 conscious responses *and* exactly S_0 subconscious responses?

d Determine the s transform for the probability density function for random variable x, where x is the time from the start of the exam until Dave makes his first conscious response which is preceded by at least one subconscious response.

e Determine the probability mass function for the total number of responses up to and including his third conscious response.

f The papers are to be collected as soon as Dave has completed exactly N responses. Determine:

i The expected number of questions he will answer correctly

ii The probability mass function for L, the number of questions he answers correctly

g Repeat part (*f*) for the case in which the exam papers are to be collected at the end of a fixed interval of T minutes.

4.14 Determine, in an efficient manner, the fourth moment of a continuous random variable described by the probability density function

$$f_x(x_0) = \begin{cases} \dfrac{4^3 x_0^2 e^{-4x_0}}{2} & x_0 \geq 0 \\ 0 & x_0 < 0 \end{cases}$$

4.15 The probability density function for L, the length of yarn purchased by any particular customer, is given by

$$f_L(L_0) = \frac{\lambda^3 L_0^2 e^{-\lambda L_0}}{2} \qquad L_0 \geq 0$$

A single dot is placed on the yarn at the mill. Determine the expected value of r, where r is the length of yarn purchased by that customer whose purchase included the dot.

4.16 A communication channel fades (degrades beyond use) in a random manner. The length of any fade is an exponential random variable with expected value λ^{-1}. The duration of the interval between the end of any fade and the start of the next fade is an Erlang random variable with PDF

$$f_t(t_0) = \frac{\mu^4 t_0^3 e^{-\mu t_0}}{3!} \qquad t_0 \geq 0$$

a If we observe the channel at a randomly selected instant, what is the probability that it will be in a fade at that time? Would you expect this answer to be equal to the fraction of all time for which the channel is degraded beyond use?

b A device can be built to make the communication system continue to operate during the first T units of time in any fade. The cost of the device goes up rapidly with T. What is the smallest value of T which will reduce by 90% the amount of time the system is out of service?

4.17 The random variable t corresponds to the interarrival time between consecutive events and is specified by the probability density function

$$f_t(t_0) = 4t_0^2 e^{-2t_0} \qquad t_0 \geq 0$$

Interarrival times are independent.

a Determine the expected value of the interarrival time x between the 11th and 13th events.

b Determine the probability density function for the interarrival time y between the 12th and 16th events.

c If we arrive at the process at a random time, determine the probability density function for the total length of the interarrival gap which we shall enter.

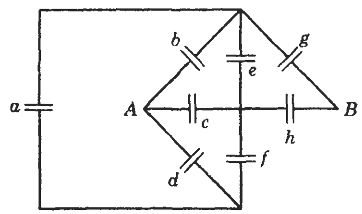
d Determine the expected value and the variance of random variable r, defined by $r = x + y$.

4.18 Bottles arrive at the Little Volcano Bottle Capper (LVBC) in a Poisson manner, with an average arrival rate of λ bottles per minute. The LVBC works instantly, but we also know that it destroys any bottles which arrive within $1/5\lambda$ minutes of the most recent successful capping operation.

a A long time after the process began, what is the probability that a randomly selected arriving bottle (marked at the bottle factory) will be destroyed?

b What is the probability that neither the randomly selected bottle nor any of the four bottles arriving immediately after it will be destroyed?

4.19 In the diagram below, each —||— represents a communication link. Under the present maintenance policy, link failures may be considered independent events, and one can assume that, at any time, the probability that any link is working properly is p.



a If we consider the system at a random time, what is the probability that:

i A total of exactly two links are operating properly?

ii Link g and exactly one other link are operating properly?

b Given that exactly six links are not operating properly at a particular time, what is the probability that A can communicate with B?

c *Under a new maintenance policy*, the system was put into operation in perfect condition at $t = 0$, and the PDF for the time until failure of any link is

$$f_t(t_0) = \lambda e^{-\lambda t_0} \qquad t_0 \geq 0$$

Link failures are still independent, but no repairs are to be made until the third failure occurs. At the time of this third failure, the system is shut down, fully serviced, and then "restarted" in perfect order. The down time for this service operation is a random variable with probability density function

$$f_x(x_0) = \mu^2 x_0 e^{-\mu x_0} \qquad x_0 \geq 0$$

i What is the probability that link g will fail before the first service operation?
ii Determine the probability density function for random variable y, the time until the first link failure after $t = 0$.
iii Determine the mean and variance and s transform for w, the time from $t = 0$ until the end of the first service operation.

4.20 The interarrival times (gaps) between the arrivals of successive events at points in time are independent random variables with PDF,

$$f_t(t_0) = \begin{cases} Kt_0(1 - t_0) & \text{if } 0 \leq t_0 \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

a What fraction of time is spent in gaps longer than the average gap?
b If we come along at a random instant after the process has been proceeding for a long time, determine
i The probability we shall see an arrival in the next (small) Δt
ii The PDF for l, the time we wait until the next arrival
c Find any $f_t(t_0)$ for which, in the notation of this problem, there would result $E(l) > E(t)$.

4.21 Two types of tubes are processed by a certain machine. Arrivals of type I tubes and of type II tubes form independent Poisson processes with average arrival rates of λ_1 and λ_2 tubes per hour, respectively. The processing time required for any type I tube, x_1, is an independent random variable with PDF

$$f_{x_1}(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

The processing time required for any type II tube, x_2, is also a uniformly distributed independent random variable

$$f_{x_2}(x) = \begin{cases} 0.5 & \text{if } 0 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

The machine can process only one tube at a time. If any tube arrives while the machine is occupied, the tube passes on to another machine station.

a Let y be the time between successive tube arrivals (regardless of type and regardless of whether the machine is free). Determine $f_y(y_0)$, $E(y)$, and σ_y^2.
b Given that a tube arrives when the machine is free, what is the probability that the tube is of type I?
c Given that the machine starts to process a tube at time T_0, what is the PDF for the time required to process the tube?
d If we inspect the machine at a random time and find it processing a tube, what is the probability that the tube we find in the machine is type I?
e Given that an idle period of the machine was exactly T hours long, what is the probability that this particular idle period was terminated by the arrival of a type I tube?

4.22 The first-order interarrival times for cars passing a checkpoint are independent random variables with PDF

$$f_t(t_0) = \begin{cases} 2e^{-2t_0} & t_0 > 0 \\ 0 & t_0 \leq 0 \end{cases}$$

where the interarrival times are measured in minutes. The successive experimental values of the durations of these first-order interarrival times are recorded on small computer cards. The recording operation occupies a negligible time period following each arrival. Each card has space for three entries. As soon as a card is filled, it is replaced by the next card.

a Determine the mean and the third moment of the first-order interarrival times.
b Given that no car has arrived in the last four minutes, determine the PMF for random variable K, the number of cars to arrive in the next six minutes.
c Determine the PDF, the expected value, and the s transform for the total time required to use up the first dozen computer cards.
d Consider the following two experiments:
i Pick a card at random from a group of completed cards and note the total time, t_i, the card was in service. Find $E(t_i)$ and $\sigma_{t_i}^2$.
ii Come to the corner at a random time. When the card in use at the time of your arrival is completed, note the total time it was in service (the time from the start of its service to its completion). Call this time t_j. Determine $E(t_j)$, and $\sigma_{t_j}^2$.
e Given that the computer card presently in use contains exactly two entries and also that it has been in service for exactly 0.5 minute, determine and sketch the PDF for the remaining time until the card is completed.