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PROFESSOR: OK. So today's lecture will be on the subject of counting. So counting, I guess, is a pretty simple affair conceptually, but it's a topic that can also get to be pretty tricky. The reason we're going to talk about counting is that there's a lot of probability problems whose solution actually reduces to successfully counting the cardinalities of various sets.

So we're going to see the basic, simplest methods that one can use to count systematically in various situations. So in contrast to previous lectures, we're not going to introduce any significant new concepts of a probabilistic nature. We're just going to use the probability tools that we already know. And we're going to apply them in situations where there's also some counting involved.

Now, today we're going to just touch the surface of this subject. There's a whole field of mathematics called combinatorics who are people who actually spend their whole lives counting more and more complicated sets. We were not going to get anywhere close to the full complexity of the field, but we'll get just enough tools that allow us to address problems of the type that one encounters in most common situations.

So the basic idea, the basic principle is something that we've already discussed. So counting methods apply in situations where we have probabilistic experiments with a finite number of outcomes and where every outcome-- every possible outcome-- has the same probability of occurring.

So we have our sample space, omega, and it's got a bunch of discrete points in there. And the cardinality of the set omega is some capital N. So, in particular, we assume that the sample points are equally likely, which means that every element of the sample space has the same probability equal to 1 over N.

And then we are interested in a subset of the sample space, call it A. And that subset consists of a number of elements. Let the cardinality of that subset be equal to little n. And then to find the probability of that set, all we need to do is to add the probabilities of the individual elements. There's little n elements, and each one has probability one over capital N. And that's the answer.

So this means that to solve problems in this context, all that we need to be able to do is to figure out the number capital N and to figure out the number little n. Now, if somebody gives you a set by just giving you a list and gives you another set, again, giving you a list, it's easy to count there element. You just count how much there is on the list.

But sometimes the sets are described in some more implicit way, and we may have to do a little bit more work. There's various tricks that are involved in counting properly. And the most common one is to-- when you consider a set of possible outcomes, to describe the construction of those possible outcomes through a sequential process.

So think of a probabilistic experiment that involves a number of stages, and in each one of the stages there's a number of possible choices that there may be. The overall experiment consists of carrying out all the stages to the end. And the number of points in the sample space is how many final outcomes there can be in this multi-stage experiment.

So in this picture we have an experiment in which of the first stage we have four choices. In the second stage, no matter what happened in the first stage, the way this is drawn we have three choices. No matter whether we ended up here, there, or there, we have three choices in the second stage.

And then there's a third stage and at least in this picture, no matter what happened in the first two stages, in the third stage we're going to have two possible choices. So how many leaves are there at the end of this tree? That's simple. It's just the product of these three numbers. The number of possible leaves that we have out there is 4 times 3 times 2. Number of choices at each stage gets multiplied, and that gives us the number of overall choices.

So this is the general rule, the general trick that we are going to use over and over. So let's apply it to some very simple problems as a warm up. How many license plates can you make if you're allowed to use three letters and then followed by four digits? At least if you're dealing with the English alphabet, you have 26 choices for the first letter. Then you have 26 choices for the second letter. And then 26 choices for the third letter.

And then we start the digits. We have 10 choices for the first digit, 10 choices for the second digit, 10 choices for the third, 10 choices for the last one. Let's make it a little more complicated, suppose that we're interested in license plates where no letter can be repeated and no digit can be repeated. So you have to use different letters, different digits. How many license plates can you make?

OK, let's choose the first letter, and we have 26 choices. Now, I'm ready to choose my second letter, how many choices do I have? I have 25, because I already used one letter. I have the 25 remaining letters to choose from. For the next letter, how many choices? Well, I used up two of my letters, so I only have 24 available.

And then we start with the digits, 10 choices for the first digit, 9 choices for the second, 8 for the third, 7 for the last one. All right. So, now, let's bring some symbols in a related problem. You are given a set that consists of n elements and you're supposed to take those n elements and put them in a sequence. That is to order them. Any possible ordering of those elements is called a permutation.

So for example, if we have the set 1, 2, 3, 4, a possible permutation is the list 2, 3, 4, 1. That's one possible permutation. And there's lots of possible permutations, of course, the question is how many are there. OK, let's think about building this permutation by choosing one at a time. Which of these elements goes into each one of these slots? How many choices for the number that goes into the first slot or the elements?

Well, we can choose any one of the available elements, so we have n choices. Let's say this element goes here, having used up that element, we're left with n minus 1 elements and we can pick any one of these and bring it into the second slot. So here we have n choices, here we're going to have n minus 1 choices, then how many we put there will have n minus 2 choices. And you go down until the end.

What happens at this point when you are to pick the last element? Well, you've used n minus of them, there's only one left in your bag. You're forced to use that one. So the last stage, you're going to have only one choice. So, basically, the number of possible permutations is the product of all integers from n down to one, or from one up to n. And there's a symbol that we use for this number, it's called n factorial.

So n factorial is the number of permutations of n objects. The number of ways that you can order n objects that are given to you. Now, a different equation. We have n elements. Let's say the elements are 1, 1,2, up to n. And it's a set. And we want to create a subset. How many possible subsets are there?

So speaking of subsets means looking at each one of the elements and deciding whether you're going to put it in to subsets or not. For example, I could choose to put 1 in, but 2 I'm not putting it in, 3 I'm not putting it in, 4 I'm putting it, and so on. So that's how you create a subset. You look at each one of the elements and you say, OK, I'm going to put it in the subset, or I'm not going to put it.

So think of these as consisting of stages. At each stage you look at one element, and you make a binary decision. Do I put it in the subset, or not? So therefore, how many subsets are there? Well, I have two choices for the first element. Am I going to put in the subset, or not? I have two choices for the next element, and so on.

For each one of the elements, we have two choices. So the overall number of choices is 2 to the power n. So, conclusion-- the number of subsets, often n element set, is 2 to the n. So in particular, if we take n equal to 1, let's check that our answer makes sense. If we have n equal to one, how many subsets does it have?

So we're dealing with a set of just one. What are the subsets? One subset is this one. Do we have other subsets of the one element set? Yes, we have the empty set. That's the second one. These are the two possible subsets of this particular set. So 2 subsets when n is equal to 1, that checks the answer.

All right. OK, so having gone so far, we can do our first example now. So we are given a die and we're going to roll it 6 times. OK, let's make some assumptions about the rolls. Let's assume that the rolls are independent, and that the die is also fair.

So this means that the probability of any particular outcome of the die rolls-- for example, so we have 6 rolls, one particular outcome could be 3,3,1,6,5. So that's one possible outcome. What's the probability of this outcome? There's probability 1/6 that this happens, 1/6 that this happens, 1/6 that this happens, and so on. So the probability that the outcome is this is 1/6 to the sixth.

What did I use to come up with this answer? I used independence, so I multiplied the probability of the first roll gives me a 2, times the probability that the second roll gives me a 3, and so on. And then I used the assumption that the die is fair, so that the probability of 2 is 1/6, the probably of 3 is 1/6, and so on.

So if I were to spell it out, it's the probability that we get the 2 in the first roll, times the probability of 3 in the second roll, times the probability of the 5 in the last roll. So by independence, I can multiply probabilities. And because the die is fair, each one of these numbers is 1/6 to the sixth.

And so the same calculation would apply no matter what numbers I would put in here. So all possible outcomes are equally likely. Let's start with this. So since all possible outcomes are equally likely to find an answer to a probability question, if we're dealing with some particular event, so the event is that all rolls give different numbers. That's our event A. And our sample space is some set capital omega. We know that the answer is going to be the cardinality of the set A, divided by the cardinality of the set omega.

So let's deal with the easy one first. How many elements are there in the sample space? How many possible outcomes are there when you roll a dice 6 times?

You have 6 choices for the first roll. You have 6 choices for the second roll and so on. So the overall number of outcomes is going to be 6 to the sixth. So number of elements in the sample space is 6 to the sixth power.

And I guess this checks with this. We have 6 to the sixth outcomes, each one has this much probability, so the overall probability is equal to one. Right? So the probability of an individual outcome is one over how many possible outcomes we have, which is this. All right.

So how about the numerator? We are interested in outcomes in which the numbers that we get are all different. So what is an outcome in which the numbers are all different? So the die has 6 faces. We roll it 6 times. We're going to get 6 different numbers. This means that we're going to exhaust all the possible numbers, but they can appear in any possible sequence.

So an outcome that makes this event happen is a list of the numbers from 1 to 6, but arranged in some arbitrary order. So the possible outcomes that make event A happen are just the permutations of the numbers from 1 to 6.

One possible outcome that makes our events to happen-- it would be this. Here we have 6 possible numbers, but any other list of this kind in which none of the numbers is repeated would also do. So number of outcomes that make the event happen is the number of permutations of 6 elements. So it's 6 factorial. And so the final answer is going to be 6 factorial divided by 6 to the sixth.

All right, so that's a typical way that's one solves problems of this kind. We know how to count certain things. For example, here we knew how to count permutations, and we used our knowledge to count the elements of the set that we need to deal with.

So now let's get to a slightly more difficult problem. We're given once more a set with n elements. We already know how many subsets that set has, but now we would be interested in subsets that have exactly k elements in them. So we start with our big set that has n elements, and we want to construct a subset that has k elements.

Out of those n I'm going to choose k and put them in there. In how many ways can I do this? More concrete way of thinking about this problem-- you have n people in some group and you want to form a committee by picking people from that group, and you want to form a committee with k people. Where k is a given number. For example, a 5 person committee. How many 5 person committees are possible if you're starting with 100 people?

So that's what we want to count. How many k element subsets are there? We don't yet know the answer, but let's give a name to it. And the name is going to be this particular symbol, which we read as n choose k. Out of n elements, we want to choose k of them.

OK. That may be a little tricky. So what we're going to do is to instead figure out a somewhat easier problem, which is going to be-- in how many ways can I pick k out of these people and puts them in a particular order? So how many possible ordered lists can I make that consist of k people? By ordered, I mean that we take those k people and we say this is the first person in the community. That's the second person in the committee. That's the third person in the committee and so on.

So in how many ways can we do this? Out of these n, we want to choose just k of them and put them in slots. One after the other. So this is pretty much like the license plate problem we solved just a little earlier.

So we have n choices for who we put as the top person in the community. We can pick anyone and have them be the first person. Then I'm going to choose the second person in the committee. I've used up 1 person. So I'm going to have n minus 1 choices here.

And now, at this stage I've used up 2 people, so I have n minus 2 choices here. And this keeps going on. Well, what is going to be the last number? Is it's n minus k? Well, not really. I'm starting subtracting numbers after the second one, so by the end I will have subtracted k minus 1. So that's how many choices I will have for the last person.

So this is the number of ways-- the product of these numbers there gives me the number of ways that I can create ordered lists consisting of k people out of the n that we started with. Now, you can do a little bit of algebra and check that this expression here is the same as that expression.

Why is this? This factorial has all the products from 1 up to n. This factorial has all the products from 1 up to n minus k. So you get cancellations. And what's left is all the products starting from the next number after here, which is this particular number.

So the number of possible ways of creating such ordered lists is n factorial divided by n minus k factorial. Now, a different way that I could make an ordered list-- instead of picking the people one at a time, I could first choose my k people who are going to be in the committee, and then put them in order. And tell them out of these k, you are the first, you are the second, you are the third.

Starting with this k people, in how many ways can I order them? That's the number of permutations. Starting with a set with k objects, in how many ways can I put them in a specific order? How many specific orders are there? That's basically the question. In how many ways can I permute these k people and arrange them.

So the number of ways that you can do this step is k factorial. So in how many ways can I start with a set with n elements, go through this process, and end up with a sorted list with k elements? By the rule that-- when we have stages, the total number of stages is how many choices we had in the first stage, times how many choices we had in the second stage. The number of ways that this process can happen is this times that.

This is a different way that that process could happen. And the number of possible of ways is this number. No matter which way we carry out that process, in the end we have the possible ways of arranging k people out of the n that we started with.

So the final answer that we get when we count should be either this, or this times that. Both are equally valid ways of counting, so both should give us the same answer. So we get this equality here. So these two expressions corresponds to two different ways of constructing ordered lists of k people starting with n people initially.

And now that we have this relation, we can send the k factorial to the denominator. And that tells us what that number, n choose k, is going to be. So this formula-- it's written here in red, because you're going to see it a zillion times until the end of the semester-- they are called the binomial coefficients. And they tell us the number of possible ways that we can create a k element subset, starting with a set that has n elements.

It's always good to do a sanity check to formulas by considering extreme cases. So let's take the case where k is equal to n. What's the right answer in this case? How many n elements subsets are there out of an element set?

Well, your subset needs to include every one. You don't have any choices. There's only one choice. It's the set itself. So the answer should be equal to 1. That's the number of n element subsets, starting with a set with n elements. Let's see if the formula gives us the right answer.

We have n factorial divided by k, which is n in our case-- n factorial. And then n minus k is 0 factorial. So if our formula is correct, we should have this equality. And what's the way to make that correct? Well, it depends what kind of meaning do we give to this symbol? How do we define zero factorial?

I guess in some ways it's arbitrary. We're going to define it in a way that makes this formula right. So the definition that we will be using is that whenever you have 0 factorial, it's going to stand for the number 1. So let's check that this is also correct, at the other extreme case.

If we let k equal to 0, what does the formula give us? It gives us, again, n factorial divided by 0 factorial times n factorial. According to our convention, this again is equal to 1. So there is one subset of our set that we started with that has zero elements. Which subset is it? It's the empty set.

So the empty set is the single subset of the set that we started with that happens to have exactly zero elements. So the formula checks in this extreme case as well. So we're comfortable using it. Now these factorials and these coefficients are really messy algebraic objects.

There's lots of beautiful identities that they satisfy, which you can prove algebraically sometimes by using induction and having cancellations happen all over the place. But it's really messy. Sometimes you can bypass those calculations by being clever and using your understanding of what these coefficients stand for.

So here's a typical example. What is the sum of those binomial coefficients? I fix n, and sum over all possible cases. So if you're an algebra genius, you're going to take this expression here, plug it in here, and then start doing algebra furiously. And half an hour later, you may get the right answer.

But now let's try to be clever. What does this really do? What does that formula count? We're considering k element subsets. That's this number. And we're considering the number of k element subsets for different choices of k.

The first term in this sum counts how many 0-element subsets we have. The next term in this sum counts how many 1-element subsets we have. The next term counts how many 2-element subsets we have. So in the end, what have we counted?

We've counted the total number of subsets. We've considered all possible cardinalities. We've counted the number of subsets of size k. We've considered all possible sizes k. The overall count is going to be the total number of subsets. And we know what this is. A couple of slides ago, we discussed that this number is equal to 2 to the n.

So, nice, clean and simple answer, which is easy to guess once you give an interpretation to the algebraic expression that you have in front of you. All right. So let's move again to sort of an example in which those binomial coefficients are going to show up.

So here's the setting-- n independent coin tosses, and each coin toss has a probability, P, of resulting in heads. So this is our probabilistic experiment. Suppose we do 6 tosses. What's the probability that we get this particular sequence of outcomes?

Because of independence, we can multiply probability. So it's going to be the probability that the first toss results in heads, times the probability that the second toss results in tails, times the probability that the third one results in tails, times probability of heads, times probability of heads, times probability of heads, which is just P to the fourth times (1 minus P) squared.

So that's the probability of this particular sequence. How about a different sequence? If I had 4 tails and 2 heads, but in a different order-- let's say if we considered this particular outcome-- would the answer be different?

We would still have P, times P, times P, times P, times (1 minus P), times (1 minus P). We would get again, the same answer. So what you observe from just this example is that, more generally, the probability of obtaining a particular sequence of heads and tails is P to a power, equal to the number of heads. So here we had 4 heads. So there's P to the fourth showing up. And then (1 minus P) to the power number of tails.

So every k head sequence-- every outcome in which we have exactly k heads, has the same probability, which is going to be P to the k, (1 minus p), to the (n minus k). This is the probability of any particular sequence that has exactly k heads. So that's the probability of a particular sequence with k heads.

So now let's ask the question, what is the probability that my experiment results in exactly k heads, but in some arbitrary order? So the heads could show up anywhere. So there's a number of different ways that this can happen. What's the overall probability that this event takes place?

So the probability of an event taking place is the sum of the probabilities of all the individual ways that the event can occur. So it's the sum of the probabilities of all the outcomes that make the event happen. The different ways that we can obtain k heads are the number of different sequences that contain exactly k heads.

We just figured out that any sequence with exactly k heads has this probability. So to do this summation, we just need to take the common probability of each individual k head sequence, times how many terms we have in this sum.

So what we're left to do now is to figure out how many k head sequences are there. How many outcomes are there in which we have exactly k heads. OK. So what are the ways that I can describe to you a sequence with k heads?

I can take my n slots that corresponds to the different tosses. I'm interested in particular sequences that have exactly k heads. So what I need to do is to choose k slots and assign heads to them. So to specify a sequence that has exactly k heads is the same thing as drawing this picture and telling you which are the k slots that happened to have heads.

So I need to choose out of those n slots, k of them, and assign them heads. In how many ways can I choose this k slots? Well, it's the question of starting with a set of n slots and choosing k slots out of the n available.

So the number of k head sequences is the same as the number of k element subsets of the set of slots that we started with, which are the n slots 1 up to n. We know what that number is.

We counted, before, the number of k element subsets, starting with a set with n elements. And we gave a symbol to that number, which is that thing, n choose k. So this is the final answer that we obtain.

So these are the so-called binomial probabilities. And they gave us the probabilities for different numbers of heads starting with a fair coin that's being tossed a number of times. This formula is correct, of course, for reasonable values of k, meaning its correct for k equals 0, 1, up to n.

If k is bigger than n, what's the probability of k heads? If k is bigger than n, there's no way to obtain k heads, so that probability is, of course, zero. So these probabilities only makes sense for the numbers k that are possible, given that we have n tosses.

And now a question similar to the one we had in the previous slide. If I write down this summation-- even worse algebra than the one in the previous slide-- what do you think this number will turn out to be? It should be 1 because this is the probability of obtaining k heads.

When we do the summation, what we're doing is we're considering the probability of 0 heads, plus the probability of 1 head, plus the probability of 2 heads, plus the probability of n heads. We've exhausted all the possibilities in our experiment. So the overall probability, when you exhaust all possibilities, must be equal to 1. So that's yet another beautiful formula that evaluates into something really simple. And if you tried to prove this identity algebraically, of course, you would have to suffer quite a bit.

So now armed with the binomial probabilities, we can do the harder problems. So let's take the same experiment again. We flip a coin independently 10 times. So these 10 tosses are independent. We flip it 10 times. We don't see the result, but somebody comes and tells us, you know, there were exactly 3 heads in the 10 tosses that you had. OK? So a certain event happened.

And now you're asked to find the probability of another event, which is that the first 2 tosses were heads. Let's call that event A. OK. So are we in the setting of discrete uniform probability laws? When we toss a coin multiple times, is it the case that all outcomes are equally likely? All sequences are equally likely?

That's the case if you have a fair coin-- that all sequences are equally likely. But if your coin is not fair, of course, heads/heads is going to have a different probability than tails/tails. If your coin is biased towards heads, then heads/heads is going to be more likely.

So we're not quite in the uniform setting. Our overall sample space, omega, does not have equally likely elements. Do we care about that? Not necessarily. All the action now happens inside the event B that we are told has occurred.

So we have our big sample space, omega. Elements of that sample space are not equally likely. We are told that a certain event B occurred. And inside that event B, we're asked to find the conditional probability that A has also occurred.

Now here's the lucky thing, inside the event B, all outcomes are equally likely. The outcomes inside B are the sequences of 10 tosses that have exactly 3 heads. Every 3-head sequence has this probability. So the elements of B are equally likely with each other.

Once we condition on the event B having occurred, what happens to the probabilities of the different outcomes inside here? Well, conditional probability laws keep the same proportions as the unconditional ones. The elements of B were equally likely when we started, so they're equally likely once we are told that B has occurred.

So to do with this problem, we need to just transport us to this smaller universe and think about what's happening in that little universe. In that little universe, all elements of B are equally likely. So to find the probability of some subset of that set, we only need to count the cardinality of B, and count the cardinality of A. So let's do that.

Number of outcomes in B-- in how many ways can we get 3 heads out of 10 tosses? That's the number we considered before, and it's 10 choose 3. This is the number of 3-head sequences when you have 10 tosses.

Now let's look at the event A. The event A is that the first 2 tosses where heads, but we're living now inside this universe B. Given that B occurred, how many elements does A have in there? In how many ways can A happen inside the B universe.

If you're told that the first 2 were heads-- sorry. So out of the outcomes in B that have 3 heads, how many start with heads/heads? Well, if it starts with heads/heads, then the only uncertainty is the location of the third head.

So we started with heads/heads, we're going to have three heads, the question is, where is that third head going to be. It has eight possibilities. So slot 1 is heads, slot 2 is heads, the third heads can be anywhere else. So there's 8 possibilities for where the third head is going to be.

OK. So what we have counted here is really the cardinality of A intersection B, which is out of the elements in B, how many of them make A happen, divided by the cardinality of B. And that gives us the answer, which is going to be 10 choose 3, divided by 8.

And I should probably redraw a little bit of the picture that they have here. The set A is not necessarily contained in B. It could also have stuff outside B. So the event that the first 2 tosses are heads can happen with a total of 3 heads, but it can also happen with a different total number of heads.

But once we are transported inside the set B, what we need to count is just this part of A. It's A intersection B and compare it with the total number of elements in the set B. Did I write it the opposite way? Yes. So this is 8 over 10 choose 3.

OK. So we're going to close with a more difficult problem now. OK. This business of n choose k has to do with starting with a set and picking a subset of k elements. Another way of thinking of that is that we start with a set with n elements and you choose a subset that has k, which means that there's n minus k that are left.

Picking a subset is the same as partitioning our set into two pieces. Now let's generalize this question and start counting partitions in general. Somebody gives you a set that has n elements. Somebody gives you also certain numbers-- n1, n2, n3, let's say, n4, where these numbers add up to n. And you're asked to partition this set into four subsets where each one of the subsets has this particular cardinality.

So you're asking to cut it into four pieces, each one having the prescribed cardinality. In how many ways can we do this partitioning? n choose k was the answer when we partitioned in two pieces, what's the answer more generally? For a concrete example of a partition, you have your 52 card deck and you deal, as in bridge, by giving 13 cards to each one of the players.

Assuming that the dealing is done fairly and with a well shuffled deck of cards, every particular partition of the 52 cards into four hands, that is four subsets of 13 each, should be equally likely. So we take the 52 cards and we partition them into subsets of 13, 13, 13, and 13. And we assume that all possible partitions, all possible ways of dealing the cards are equally likely. So we are again in a setting where we can use counting, because all the possible outcomes are equally likely.

So an outcome of the experiment is the hands that each player ends up getting. And when you get the cards in your hands, it doesn't matter in which order that you got them. It only matters what cards you have on you. So it only matters which subset of the cards you got.

All right. So what's the cardinality of the sample space in this experiment? So let's do it for the concrete numbers that we have for the problem of partitioning 52 cards. So think of dealing as follows-- you shuffle the deck perfectly, and then you take the top 13 cards and give them to one person. In how many possible hands are there for that person?

Out of the 52 cards, I choose 13 at random and give them to the first person. Having done that, what happens next? I'm left with 39 cards. And out of those 39 cards, I pick 13 of them and give them to the second person. Now I'm left with 26 cards.

Out of those 26, I choose 13, give them to the third person. And for the last person there isn't really any choice. Out of the 13, I have to give that person all 13. And that number is just equal to 1. So we don't care about it.

All right. So next thing you do is to write down the formulas for these numbers. So, for example, here you would have 52 factorial, divided by 13 factorial, times 39 factorial, and you continue. And then there are nice cancellations that happen.

This 39 factorial is going to cancel the 39 factorial that comes from there, and so on. After you do the cancellations and all the algebra, you're left with this particular answer, which is the number of possible partitions of 52 cards into four players where each player gets exactly 13 hands.

If you were to generalize this formula to the setting that we have here, the more general formula is-- you have n factorial, where n is the number of objects that you are distributing, divided by the product of the factorials of the-- OK, here I'm doing it for the case where we split it into four sets. So that would be the answer when we partition a set into four subsets of prescribed cardinalities.

And you can guess how that formula would generalize if you want to split it into five sets or six sets. OK. So far we just figured out the size of the sample space. Now we need to look at our event, which is the event that each player gets an ace, let's call that event A. In how many ways can that event happens? How many possible hands are there in which every player has exactly one ace?

So I need to think about the sequential process by which I distribute the cards so that everybody gets exactly one ace, and then try to think in how many ways can that sequential process happen. So one way of making sure that everybody gets exactly one ace is the following-- I take the four aces and I distribute them randomly to the four players, but making sure that each one gets exactly one ace. In how many ways can that happen?

I take the ace of spades and I send it to a random person out of the four. So there's 4 choices for this. Then I'm left with 3 aces to distribute. That person already gotten an ace. I take the next ace, and I give it to one of the 3 people remaining. So there's 3 choices for how to do that.

And then for the next ace, there's 2 people who have not yet gotten an ace, and they give it randomly to one of them. So these are the possible ways of distributing for the 4 aces, so that each person gets exactly one. It's actually the same as this problem.

Starting with a set of four things, in how many ways can I partition them into four subsets where the first set has one element, the second has one element, the third one has another element, and so on. So it agrees with that formula by giving us 4 factorial.

OK. So there are different ways of distributing the aces. And then there's different ways of distributing the remaining 48 cards. How many ways are there? Well, I have 48 cards that I'm going to distribute to four players by giving 12 cards to each one. It's exactly the same question as the one we had here, except that now it's 48 cards, 12 to each person. And that gives us this particular count.

So putting all that together gives us the different ways that we can distribute the cards to the four players so that each one gets exactly one ace. The number of possible ways is going to be this four factorial, coming from here, times this number-- this gives us the number of ways that the event of interest can happen-- and then the denominator is the cardinality of our sample space, which is this number.

So this looks like a horrible mess. It turns out that this expression does simplify to something really, really simple. And if you look at the textbook for this problem, you will see an alternative derivation that gives you a short cut to the same numerical answer. All right. So that basically concludes chapter one. From next time we're going to consider introducing random variables and make the subject even more interesting.