Massachusetts Institute of Technology

Department of Electrical Engineering & Computer Science

6.041/6.431: Probabilistic Systems Analysis

(Fall 2010)

Tutorial 7: Solutions

1. (a) For each round, the probability that both Alice and Bob have a loss is $\frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$. Let random variable X represent the total number of rounds played until the first time where they both have a loss. Then X is a geometric random variable with parameter $p = 1/9$ and has the following PMF.

$$p_X(x) = (1-p)^{x-1}p = \left(\frac{8}{9}\right)^{x-1}\left(\frac{1}{9}\right), \qquad x = 1, 2, \ldots$$

(b) First, consider the number of games, K_3 Bob played until his third loss. Random variable K_3 is a Pascal random variable and has the following PMF.

$$p_{K_3}(k) = \binom{k-1}{3-1}\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^{k-3} \qquad k = 3, 4, 5, \cdots$$

In this question, we are interested in another random variable Z defined as the time at which Bob has his third loss. Note that $Z = 2K_3$. By changing variables, we obtain

$$p_Z(z) = \binom{\frac{z}{2}-1}{3-1}\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^{\frac{z}{2}-3} \qquad z = 6, 8, 10, \cdots$$

(c) Let A be the event that Alice wins, and Let B be the event that Bob wins. The event $A \cup B$ is then the event that either A wins or B wins or both A and B win, and the event $A \cap B$ is the event that both A and B win. Suppose we observe this gambling process, and let U be a random variable indicating the number of rounds we see until at least one of them wins. Random variable U is a geometric random variable with parameter $p = P(A \cup B) = 1 - \frac{1}{3} \cdot \frac{1}{3}$.

Consider another random variable V representing the number of additional rounds we have to observe until the other wins. If both Alice and Bob win at the Uth round, then $V = 0$. This occurs with probability $P(A \cap B | A \cup B) = \frac{\frac{2}{3}\frac{2}{3}}{\frac{8}{9}}$. If Alice wins the Uth round, then the time V until Bob wins is a geometric random variable with parameter $p = 1/2 + 1/6 = 2/3$. This occurs with probability $P(A \cap B^c | A \cup B) = \frac{\frac{1}{3}\frac{2}{3}}{\frac{8}{9}}$. Likewise, if Bob wins the Uth round, then the time V until Alice wins is a geometric random variable with parameter $p = 1/2 + 1/6 = 2/3$. This occurs with probability $P(B \cap A^c | A \cup B) = \frac{\frac{1}{3}\frac{2}{3}}{\frac{8}{9}}$.The number of rounds until each one of them has won at least once, N is

$$N = U + V$$

The expectation of N is then:

$$\begin{aligned}
E[N] &= E[U] + E[V] \\
&= \frac{1}{\frac{8}{9}} + 0 \cdot P(A \cap B | A \cup B) + \frac{1}{\frac{2}{3}}P(A | A \cup B) + \frac{1}{\frac{2}{3}}P(B | A \cup B) \\
&= 9/8 + \frac{3}{2}\frac{\frac{1}{3}\frac{2}{3}}{\frac{8}{9}} + \frac{3}{2}\frac{\frac{1}{3}\frac{2}{3}}{\frac{8}{9}} \\
&= 15/8
\end{aligned}$$

There is another approach to this problem. Consider the following partition.

A_1: both win first round
A_2:Only Alice wins first round
A_3: Only Bob wins first round
A_4: both lose first round

Event A_1 occurs with probability $\frac{2}{3} \cdot \frac{2}{3}$. Event A_2 occurs with probability $\frac{2}{3} \cdot \frac{1}{3}$. Event A_3 occurs with probability $\frac{1}{3} \cdot \frac{2}{3}$. Event A_4 occurs with probability $\frac{1}{3} \cdot \frac{1}{3}$. When event A_2 (A_3) occurs, the distribution on the time until Bob (Alice) wins is a geometric random variable with mean $\frac{1}{\frac{2}{3}}$. When event A_4 occurs, the additional time until Alice and Bob win is distributed identically to that at time 0 by the fresh-start property. By the total expectation theorem,

$$
\begin{aligned}
E[N] &= E[N|A_1]P(A_1) + E[N|A_2]P(A_2) + E[N|A_3]P(A_3)E[N] + E[N|A_4]P(A_4) \\
&= 1 \cdot \left(\frac{2}{3} \cdot \frac{2}{3}\right) + \left(1 + \frac{1}{\frac{2}{3}}\right) \cdot + \left(\frac{1}{3} \cdot \frac{2}{3}\right) + \left(\frac{1}{3} \cdot \frac{2}{3}\right) + (1 + E[N]) \cdot \left(\frac{1}{3} \cdot \frac{1}{3}\right)
\end{aligned}
$$

Solving for $E[N]$, we get $E[N] = \frac{15}{8}$.

2. Problem 6.6, page 328 in text. See text for solutions.

3. (a) The number of trains arriving on days 1, 2, and 3 is independent of the number of trains arriving on day 0. Let N denote the total number of trains that arrive on days 1, 2, and 3. Then N is a Poisson random variable with parameter $3\lambda = 9$, and we have

$$
\begin{aligned}
P(\text{no train on days 1,2,3} \mid \text{one train on day 1}) &= P(\text{no train on days 1,2,3}) \\
&= P(N = 0) \\
&= \frac{e^{-9} 9^0}{0!} \\
&= e^{-9}.
\end{aligned}
$$

(b) The event that the next arrival is more than three days after the train arrival on day 0 is the same as the event that there are zero arrivals in the three days after the train arrival on day 0. Therefore the required probability is the same as that found in part (a), namely, e^{-9}.

(c) The number of trains arriving in the first 2 days is independent of the number of trains arriving on day 4. Therefore, we have

$$
\begin{aligned}
P(\text{0 trains in first 2 days and 4 trains on day 4}) &= P(\text{0 trains in first 2 days}) \cdot P(\text{4 trains on day 4}) \\
&= e^{-2\lambda} \frac{(2\lambda)^0}{0!} \cdot e^{-\lambda} \frac{\lambda^4}{4!} \\
&= e^{-9} \frac{3^4}{4!}.
\end{aligned}
$$

(d) The event that it takes more than 2 days for the 5th arrival is equivalent to the event that there are at most 4 arrivals in the first 2 days. Therefore the required probability is equal to

$$\begin{aligned}
\sum_{k=0}^{4} P(\text{exactly } k \text{ arrivals in first 2 days}) &= \sum_{k=0}^{4} e^{-2\lambda}\frac{(2\lambda)^k}{k!} \\
&= e^{-2\lambda}\left(\frac{(2\lambda)^0}{0!} + \frac{(2\lambda)^1}{1!} + \frac{(2\lambda)^2}{2!} + \frac{(2\lambda)^3}{3!} + \frac{(2\lambda)^4}{4!}\right) \\
&= e^{-6}(1 + 6 + 18 + 36 + 54) \\
&= 115e^{-6}\,.
\end{aligned}$$