6.041 Fall 2009 Quiz 2
Tuesday, November 3, 7:30 - 9:30 PM.

DO NOT TURN THIS PAGE OVER UNTIL YOU ARE TOLD TO DO SO

Name: ______________________

Recitation Instructor: ______________________

TA: ______________________

Question	Score	Out of
1		2
2 (a)		7
2 (b)		7
2 (c)		7
2 (d)		7
2 (e)		7
2 (f)		7
3		10
4 (a i)		5
4 (a ii)		5
4 (b)		7
4 (c)		8
5 (a)		7
5 (b)		7
5 (c)		7
Your Grade		100

- This quiz has 5 problems, worth a total of 100 points.
- **When giving a formula for a PDF, make sure to specify the range over which the formula holds.**
- Please make sure to return the entire exam booklet intact.
- Write your solutions in this quiz booklet, only solutions in this quiz booklet will be graded. Be neat! You will not get credit if we can't read it.
- You are allowed 2 two-sided, handwritten, formulae sheets. Calculators not allowed.
- You may give an answer in the form of an arithmetic expression (sums, products, ratios, factorials) of numbers that could be evaluated using a calculator. Expressions like $\binom{8}{3}$ or $\sum_{k=0}^{5}(1/2)^k$ are also fine.
- You have 2 hrs. to complete the quiz.
- Graded quizzes will be returned in recitation on Thursday 11/5.

	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767

The standard normal table. The entries in this table provide the numerical values of $\Phi(y) = \mathbf{P}(Y \leq y)$, where Y is a standard normal random variable, for y between 0 and 1.99. For example, to find $\Phi(1.71)$, we look at the row corresponding to 1.7 and the column corresponding to 0.01, so that $\Phi(1.71) = .9564$. When y is negative, the value of $\Phi(y)$ can be found using the formula $\Phi(y) = 1 - \Phi(-y)$.

Problem 2. (42 points)
The random variable X is exponential with parameter 1. Given the value x of X, the random variable Y is exponential with parameter equal to x (and mean $1/x$).

Note: Some useful integrals, for $\lambda > 0$:

$$\int_0^\infty xe^{-\lambda x}\,dx = \frac{1}{\lambda^2}, \qquad \int_0^\infty x^2 e^{-\lambda x}\,dx = \frac{2}{\lambda^3}.$$

(a) (7 points) Find the joint PDF of X and Y.

(b) (7 points) Find the marginal PDF of Y.

(c) (7 points) Find the conditional PDF of X, given that $Y = 2$.

(d) (7 points) Find the conditional expectation of X, given that $Y = 2$.

(e) (7 points) Find the conditional PDF of Y, given that $X = 2$ and $Y \geq 3$.

(f) (7 points) Find the PDF of e^{2X}.

Problem 3. (10 points)
For the following questions, mark the correct answer. If you get it right, you receive 5 points for that question. You receive no credit if you get it wrong. A justification is not required and will not be taken into account.

Let X and Y be continuous random variables. Let N be a discrete random variable.

(a) (5 points) The quantity $\mathbf{E}[X \mid Y]$ is always:

(i) A number.
(ii) A discrete random variable.
(iii) A continuous random variable.
(iv) Not enough information to choose between (i)-(iii).

(b) (5 points) The quantity $\mathbf{E}[\,\mathbf{E}[X \mid Y, N] \mid N]$ is always:

(i) A number.
(ii) A discrete random variable.
(iii) A continuous random variable.
(iv) Not enough information to choose between (i)-(iii).

Problem 4. (25 points)
The probability of obtaining heads in a single flip of a certain coin is itself a random variable, denoted by Q, which is uniformly distributed in $[0, 1]$. Let $X = 1$ if the coin flip results in heads, and $X = 0$ if the coin flip results in tails.

(a) (i) (5 points) Find the mean of X.
(ii) (5 points) Find the variance of X.

(b) (7 points) Find the covariance of X and Q.

(c) (8 points) Find the conditional PDF of Q given that $X = 1$.

Problem 5. (21 points)
Let X and Y be **independent continuous** random variables with marginal PDFs f_X and f_Y, and marginal CDFs F_X and F_Y, respectively. Let

$$S = \min\{X, Y\}, \qquad L = \max\{X, Y\}.$$

(a) (7 points) If X and Y are standard normal, find the probability that $S \geq 1$.

(b) (7 points) Fix some s and ℓ with $s \leq \ell$. Give a formula for

$$\mathbf{P}(s \leq S \text{ and } L \leq \ell)$$

involving F_X and F_Y, and no integrals.

(c) (7 points) Assume that $s \leq s + \delta \leq \ell$. Give a formula for

$$\mathbf{P}(s \leq S \leq s + \delta,\ \ell \leq L \leq \ell + \delta),$$

as an integral involving f_X and f_Y.

Each question is repeated in the following pages. Please write your answer on the appropriate page.

Problem 2. (42 points)
The random variable X is exponential with parameter 1. Given the value x of X, the random variable Y is exponential with parameter equal to x (and mean $1/x$).

Note: Some useful integrals, for $\lambda > 0$:

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(a) (7 points) Find the joint PDF of X and Y.

(b) (7 points) Find the marginal PDF of Y.

(c) (7 points) Find the conditional PDF of X, given that $Y = 2$.

(d) (7 points) Find the conditional expectation of X, given that $Y = 2$.

(e) (7 points) Find the conditional PDF of Y, given that $X = 2$ and $Y \geq 3$.

(f) (7 points) Find the PDF of e^{2X}.

Problem 3. (10 points)
For the following questions, mark the correct answer. If you get it right, you receive 5 points for that question. You receive no credit if you get it wrong. A justification is not required and will not be taken into account.

Let X and Y be continuous random variables. Let N be a discrete random variable.

(a) (5 points) The quantity $\mathbf{E}[X \mid Y]$ is always:

- (i) A number.
- (ii) A discrete random variable.
- (iii) A continuous random variable.
- (iv) Not enough information to choose between (i)-(iii).

(b) (5 points) The quantity $\mathbf{E}[\,\mathbf{E}[X \mid Y, N] \mid N]$ is always:

- (i) A number.
- (ii) A discrete random variable.
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involving F_X and F_Y, and no integrals.

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