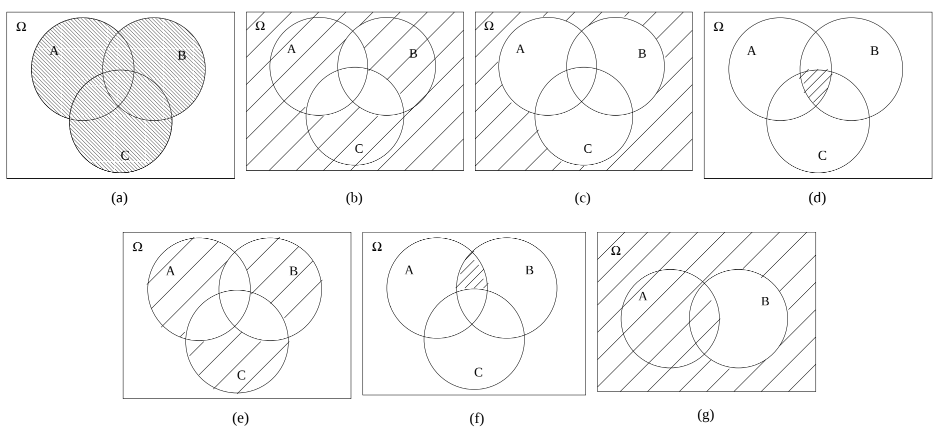
MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering & Computer Science

6.041/6.431: Probabilistic Systems Analysis

(Fall 2010)

Problem Set 1: Solutions
Due: September 15, 2010

1. (a) $A \cup B \cup C$
   (b) $(A \cap B^c \cap C^c) \cup (A^c \cap B \cap C^c) \cup (A^c \cap B^c \cap C) \cup (A^c \cap B^c \cap C^c)$
   (c) $(A \cup B \cup C)^c = A^c \cap B^c \cap C^c$
   (d) $A \cap B \cap C$
   (e) $(A \cap B^c \cap C^c) \cup (A^c \cap B \cap C^c) \cup (A^c \cap B^c \cap C)$
   (f) $A \cap B \cap C^c$
   (g) $A \cup (A^c \cap B^c)$



2. Since all outcomes are equally likely we apply the discrete uniform probability law to solve the problem. To solve for any event we simply count the number of elements in the event and divide by the total number of elements in the sample space.

   There are 2 possible outcomes for each flip, and 3 flips. Thus there are $2^3 = 8$ elements (or sequences) in the sample space.

   (a) Any sequence has probability of 1/8. Therefore $\mathbf{P}(\{H, H, H\}) = \boxed{1/8}$.

   (b) This is still a single sequence, thus $\mathbf{P}(\{H, T, H\}) = \boxed{1/8}$.

   (c) The event of interest has 3 unique sequences, thus $\mathbf{P}(\{HHT, HTH, THH\}) = \boxed{3/8}$.

   (d) The sequences where there are more heads than tails are $A : \{HHH, HHT, HTH, THH\}$. 4 unique sequences gives us $\mathbf{P}(A) = \boxed{1/2}$.

3. The easiest way to solve this problem is to make a table of some sort, similar to the one below.

Die 1	Die 2	Sum	**P**(Sum)
1	1	2	2p
1	2	3	3p
1	3	4	4p
1	4	5	5p
2	1	3	3p
2	2	4	4p
2	3	5	5p
2	4	6	6p
3	1	4	4p
3	2	5	5p
3	3	6	6p
3	4	7	7p
4	1	5	5p
4	2	6	6p
4	3	7	7p
4	4	8	8p
		Total	80p

$$\mathbf{P}(\text{All outcomes}) = 80p \text{ (Total from the table)}$$

and therefore

$$p = \frac{1}{80}$$

(a)

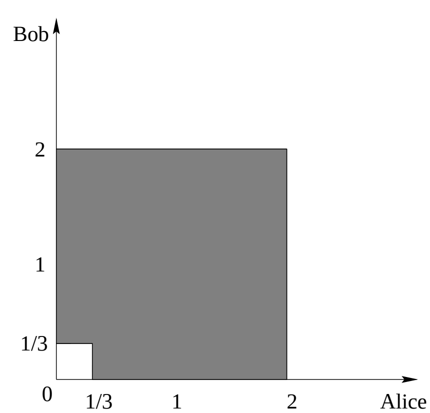
$$\mathbf{P}(\text{Even sum}) = 2p + 4p + 4p + 6p + 4p + 6p + 6p + 8p = 40p = \boxed{1/2}$$

(b)

$$\mathbf{P}(\text{Rolling a 2 and a 3}) = \mathbf{P}(2,3) + \mathbf{P}(3,2) = 5p + 5p = 10p = \boxed{1/8}$$

4. **P(B)**

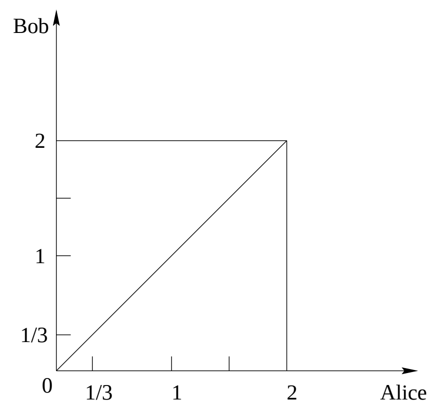
The shaded area in the following figure is the union of Alice's pick being greater than 1/3 and Bob's pick being greater than 1/3.



$$\begin{aligned}
\mathbf{P}(B) &= 1 - \mathbf{P}(\text{both numbers are smaller than } 1/3) \\
&= 1 - \frac{\text{area of small square}}{\text{total sample area}} \\
&= 1 - \frac{(1/3)(1/3)}{4} = 1 - \frac{1}{36} = \boxed{35/36}
\end{aligned}$$

P(C)

In the following figure, the diagonal line represents the set of points where the two selected numbers are equal.

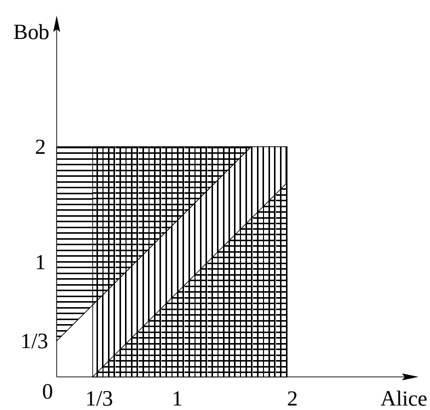


The line has an area of 0. Thus,

$$\mathbf{P}(C) = \frac{\text{area of line}}{\text{total sample area}} = \frac{0}{4} = \boxed{0}$$

P($A \cap D$)

Overlapping the diagrams we would get for $\mathbf{P}(A)$ and $\mathbf{P}(D)$,



$$\begin{aligned}\mathbf{P}(A \cap D) &= \frac{\text{double shaded area}}{\text{total sample area}} \\ &= \frac{(5/3)(5/3)(1/2) + (4/3)(4/3)(1/2)}{4} = \frac{25/18 + 16/18}{4} = \boxed{41/72}\end{aligned}$$

5. (a) The probability of Mike scoring 50 points is proportional to the area of the inner disk. Hence, it is equal to $\alpha\pi R^2 = \alpha\pi$, where α is a constant to be determined.

   Since the probability of landing the dart on the board is equal to one, $\alpha\pi 10^2 = 1$, which implies that $\alpha = 1/(100\pi)$.

   Therefore, the probability that Mike scores 50 points is equal to $\pi/(100\pi) = \boxed{0.01}$

   (b) In order to score exactly 30 points, Mike needs to place the dart between 1 and 3 inches from the origin. An easy way to compute this probability is to look first at that of scoring *more* than 30 points, which is equal to $\alpha\pi 3^2 = 0.09$.

   Next, since the 30 points ring is disjoint from the 50 points disc, probability of scoring more than 30 points is equal to the probability of scoring 50 points plus that of scoring exactly 30 points. Hence, the probability of Mike scoring exactly 30 points is equal to $0.09 - 0.01 = \boxed{0.08}$

   (c) For the part (a) question. The probability of John scoring 50 points is equal to the probability of throwing in the right half of the board and scoring 50 points plus that of throwing in the left half and scoring 50 points.

   The first term in the sum is proportional to the area of the right half of the inner disk and is equal to $\alpha\pi R^2/2 = \alpha\pi/2$, where α is a constant to be determined.

   Similarly, the probability of him throwing in the left half of the board and scoring 50 points is equal to $\beta\pi/2$, where β is a constant (not necessarily equal to α).

   In order to determine α and β, let us compute the probability of throwing the dart in the right half of the board. This probability is equal to

   $$\alpha\pi R^2/2 = \alpha\pi 10^2/2 = \alpha 50\pi.$$

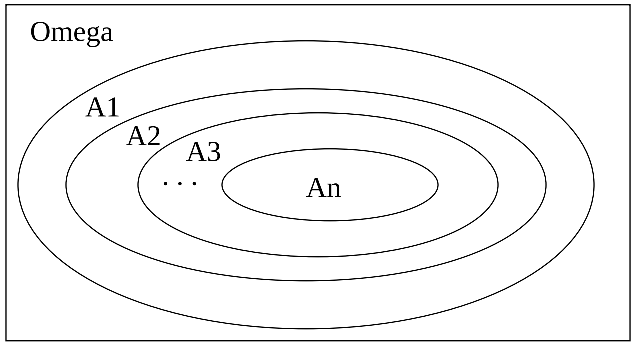
   Since that probability is equal to $2/3$, $\alpha = 1/(75\pi)$. In a similar fashion, β can be determined to be $1/(150\pi)$. Consequently, the total probability is equal to $1/150 + 1/300 = \boxed{0.01}$

   For the part (b), The probability of scoring exactly 30 points is equal to that of scoring more than 30 points minus that of scoring exactly 50. By applying the same type of analysis as in (b) above, the probability is found to be equal to $\boxed{0.08}$

   These numbers suggest that John and Mike have similar skills, and are equally likely to win the game. The fact that Mike's better control (or worst, depending on how you look at it) of the direction of his throw does not increase his chances of winning can be explained by the observation that both players' control over the distance from the origin is identical.

6. See the textbook, Problem 1.11 page 55, which proves the general version of Bonferroni's inequality.

G1†. (a) If we define $A_n = [a_n, b_n]$ for all n, it is easy to see that the sequence $A_1, A_2, \ldots$ is "monotonically decreasing," i.e., $A_{n+1} \subset A_n$ for all n:



Furthermore, $\cap_n^\infty A_n = [a, b]$.

By the continuity property of probabilities (see Problem 1.13, page 56 of the text),

$$\lim_{n\to\infty} \mathbf{P}([a_n, b_n]) = \mathbf{P}([a, b]).$$

(b) No. Consider the following example. Let $a_n = a + \frac{1}{n}$, $b_n = b - \frac{1}{n}$ for all n. Then $\{a_n\}$ is a decreasing sequence that converges to a, and $\{b_n\}$ is an increasing sequence that converges to b. If we define a probability law that places non-zero probability only on points a and b, then $\lim_{n\to\infty} \mathbf{P}([a_n, b_n]) = 0$, but $\mathbf{P}([a, b]) = 1$.

This example is closely related to the continuity property of probabilities. In this case, if we define $A_n = [a_n, b_n]$, then $A_1, A_2, \ldots$ is "monotonically increasing," i.e., $A_n \subset A_{n+1}$, but $A = (\cup_n^\infty A_n) = (a, b)$, which is an open interval whose probability is 0 under our probability law.